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GLAUBERMAN-WATANABE CORRESPONDING p-BLOCKS OF FINITE GROUPS WITH NORMAL DEFECT GROUPS ARE MORITA EQUIVALENT

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Abstract. Let G be a finite group and let A be a solvable finite group that acts on G such that the orders of G and A are relatively prime. Let b be a p-block of G with normal defect group D such that A stabilizes b and $D \leq C_G(A)$. Then there is a Morita equivalence between the block b and its Watanabe correspondent block $W(b)$ of $C_G(A)$ given by a bimodule M with vertex ΔD and trivial source that on the character level induces the Glauberman correspondence (and which is an isotypy by a theorem of Watanabe).

Introduction and statements

This study was suggested by the work of S. Koshitani and G. Michler in [13].

The Theory of Blocks of Finite Groups was introduced and significantly developed by R. Brauer in [1] and [2]. Clearly Brauer's First Main Theorem (in [1]) underlines the importance of studying a block B of a finite group G with normal defect group D. In [16], W.F. Reynolds, using Clifford Theory, presented a deep analysis of the character theory of such a block B. In [15], B. Külshammer, using fundamental Clifford theoretic methods of E.C. Dade, showed that, in the context of a standard "p-modular system" $(\mathcal{K}, \mathcal{O}, k = \mathcal{O}/J(\mathcal{O}))$, the block algebra $\mathcal{A}$ over $\mathcal{O}$ of such a block B is $\mathcal{O}$-algebra isomorphic to a full matrix algebra over a twisted group algebra $\mathcal{B}$ over $\mathcal{O}$ of the group $\mathcal{N}$ of [16]. In our main theorem (Theorem 2), we use the approach of [8] and [15] to demonstrate that the Morita equivalence between $\mathcal{A}$ and $\mathcal{B}$ is given by a bimodule with a diagonal vertex and trivial source. Moreover we are able to incorporate the Glauberman-Watanabe context into the analysis to demonstrate that there is such a Morita equivalence that gives the Glauberman correspondence on the character level. This analysis also extends the character theoretic analysis of W.F. Reynolds in [16].

Throughout this section, G will denote a finite group and A will denote a solvable finite group that acts on G on the right and such that $(|G|, |A|) = 1$. Let $C = C_G(A)$ and let $\mathrm{Irr}(G)$ and $\mathrm{Irr}(G)^A$ denote the sets of ordinary irreducible and A-invariant ordinary irreducible characters of G, respectively. In [6], G. Glauberman produced a bijection $\pi(G, A) : \mathrm{Irr}(G)^A \to \mathrm{Irr}(C)$ satisfying natural basic properties (cf. [10, Theorem 13.1]).

Let p be a prime and let $\mathcal{O}$ be a complete discrete valuation ring of characteristic zero such that $k = \mathcal{O}/J(\mathcal{O})$ is an algebraically closed field of prime characteristic

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p and such that if $\mathcal{K}$ is the fraction field of $\mathcal{O}$, then $(\mathcal{K}, \mathcal{O}, k)$ is "big enough" for all subgroups of $G \rtimes A$. As is standard, the natural ring epimorphism $- : \mathcal{O} \to \mathcal{O}/J(\mathcal{O}) = k$ induces a natural $\mathcal{O}$-algebra epimorphism $- : \mathcal{A} \to \bar{\mathcal{A}} = \mathcal{A}/J(\mathcal{O})\mathcal{A}$ for any $\mathcal{O}$-algebra $\mathcal{A}$.

Let $B\ell(G)$ and $B\ell(G)^A$ denote the set of p-blocks of G and of A-stable p-blocks of G, resp., in this context. Let $b \in B\ell(G)^A$ with defect group D such that $D \leq C = C_G(A)$. In [19, Theorems 1 and 2] A. Watanabe proved that then $\mathrm{Irr}(b) \subseteq \mathrm{Irr}(G)^A$ and $\{\pi(G, A)(\chi) \mid \chi \in \mathrm{Irr}(b)\}$ is the set of ordinary irreducible characters of a p-block of C with defect group D which we shall denote by $W(b)$. Moreover, she also proved that the Glauberman correspondence induces an isotypy between b and $W(b)$ ([19, Theorem 2]).

Clearly A acts on $N_G(D)$ and the Brauer correspondent p-block $Br_D(b)$ of b is an A-stable p-block of $N_G(D)$ with defect group D. Also the Brauer correspondent $Br_D(W(b))$ is a p-block of $N_C(D)$ with defect group D and $W(Br_D(b))$ is also a p-block of $N_C(D)$ with defect group D.

For the convenience of the reader, we include a reformulation and an alternate proof of a result of S. Koshitani and G. Michler ([14, Theorem 2.12]) that links our main result presented below with the general Glauberman-Watanabe correspondence described above.

Theorem 1. $W(Br_D(b)) = Br_D(W(b))$.

For our main result, we also assume that the defect group D of b is normal in G and hence normal in C. In the main result (Proposition 3.3) of [14], S. Koshitani and G. Michler demonstrated that, in this case, the k-algebras $kG\bar{b}$ and $kC\overline{W(b)}$ are Morita equivalent. The first part of our main result (Theorem 2), just below, stating that there is a Morita equivalence "over $\mathcal{O}$" (that induces a Morita equivalence "over k") is essentially in the paper [14]. In fact, S. Koshitani observed this fact in [13]. Moreover this "lifting from k to $\mathcal{O}$" is to be expected in view of the work of L. Puig for blocks with a normal defect group (cf. [18, Section 45 and Proposition 38.8]). Since "coefficients in $\mathcal{O}$" provides the connection in finite group representation theory between the characteristic p and classical characteristic 0 representation theories, such investigations are very important.

Our main result is:

Theorem 2. *In this situation, there is a Morita equivalence between the block algebras $(\mathcal{O}G)b$ and $(\mathcal{O}C)W(b)$ given by an indecomposable $(\mathcal{O}G)b$-mod-$(\mathcal{O}C)W(b)$ bimodule M with the following properties:*

(i) *when viewed as an $\mathcal{O}(G \times C)$-module, M has $\Delta D = \{(u, u) \mid u \in D\}$ as a vertex and a trivial $\mathcal{O}\Delta D$-source; and*

(ii) *the bijection between the sets of ordinary irreducible characters $\mathrm{Irr}_{\mathcal{K}}(G, b)$ and $\mathrm{Irr}_{\mathcal{K}}(C, W(b))$ induced by the Morita equivalence given by M is precisely the Glauberman correspondence.*

Theorems 1 and 2 immediately yield:

Corollary 3. *In the Glauberman-Watanabe context $(G, A, b \in B\ell(G)^A$, $C = C_G(A)$, $W(b)$, $D \leq C)$, the Brauer correspondent blocks $(N_G(D), Br_D(b))$ and $(N_C(D), Br_D(W(b)))$ are Morita equivalent with an equivalence given by an indecomposable $\mathcal{O}N_G(D)Br_D(b)$-mod-$(\mathcal{O}N_C(D)Br_D(W(b))$ bimodule M such that, when viewed as an $\mathcal{O}(N_G(D) \times N_C(D))$-module, M has $\Delta D = \{(u, u) \mid u \in D\}$*

as a vertex and a trivial $\mathcal{O}(\Delta D)$-source. Moveover the bijection between the sets of ordinary irreducible characters $\mathrm{Irr}_{\mathcal{K}}(N_G(D), Br_D(b))$ *and* $\mathrm{Irr}_{\mathcal{K}}(N_C(D), Br_D(W(b))$ *induced by the Morita equivalence is precisely the Glauberman correspondence.*

Clearly the main result (Proposition 3.3) of [14] is a consequence of Theorem 2.

Our notation is standard and tends to follow the notation of [5], [9] and [10].

In particular, all rings are assumed to have identities.

Section 1 is comprised of a variety of results that are required for our proofs of Theorems 1 and 2 which are presented in Section 2.

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1. Preliminary results

Let m be a positive integer and let Q be an abelian group. By definition, Q is said to be m-divisible if for each $x \in Q$ there is a $y \in Q$ such that $y^m = x$, in which case Q is also n-divisible whenever $n \mid m$.

The proof of [10, Lemma 11.14] is readily adapted to prove:

Lemma 1.1. *Let Q be a subgroup of an abelian group M such that $|M/Q|$ is finite. Assume also that Q is $|M/Q|$-divisible. Then Q is complemented in M.*

Let M be an abelian group on which a finite group G of order m acts trivially.

The following results are well known and easy to verify.

Lemma 1.2. *Let $c \in Z^2(G, M)$. Then*

(a) *$c(g, 1) = c(1, 1) = c(1, g)$ for all $g \in G$;*

(b) *$c(g, g^{-1}) = c(g^{-1}, g)$ for all $g \in G$;*

(c) *if $c' : G \times G \to M$ is defined by $c'(g, h) = c(g, h)c(1, 1)^{-1}$ for all $(g, h) \in G \times G$, then $c' \in Z^2(G, M)$ and $c'(g, 1) = c'(1, g) = 1$ for all $g \in G$;*

(d) *assume that $c(g, 1) = 1 = c(1, g)$ for all $g \in G$ and let $Z = \langle c(g, h) | (g, h) \in G \times G\rangle$, so that $Z \leq M$. Let $\hat{G} = Z \times G$ and define a multiplication on $\hat{G}$ by*

$$(z_1, g_1)(z_2, g_2) = (z_1 z_2 c(g_1, g_2), g_1 g_2)$$

for all $z_1, z_2 \in Z$ and all $g_1, g_2 \in G$. Then $\hat{G}$ is a group with identity $(1, 1)$ and $(z, g)^{-1} = (z^{-1} c(g, g^{-1})^{-1}, g^{-1})$ for all $z \in Z$ and $g \in G$. Also

$$1 \to Z \xrightarrow{i} \hat{G} \xrightarrow{\pi_1} G \to 1 \tag{1.1}$$

is a short exact sequence of groups where π_1 is the first component projection and $i : Z \to G$ is defined by $z \mapsto (z, 1)$ for all $z \in Z$, so that $i(Z) \leq Z(\hat{G})$. Suppose also that the group E acts on G on the right and $c(g^e, h^e) = c(g, h)$ for all $g, h \in G$ and all $e \in E$. Let E act trivially on the right on Z and diagonally on $Z \times G$. Then E acts on the group $\hat{G} = Z \times G$ and (1.1) is a short exact sequence of E-groups.

Remark 1.3. Suppose that $(\mathcal{K}, \mathcal{O}, k)$ is a p-modular system for the finite group G, that G acts trivially on $\mathcal{O}^\times$, and that $c \in Z^2(G, \mathcal{O}^\times)$ with $c(g, 1) = 1 = c(1, g)$ for all $g \in G$. Let $\mathcal{A} = \oplus_{g \in G} \mathcal{O}\tau_g$ be an associated twisted group $\mathcal{O}$-algebra over G, where $\tau_g \tau_h = c(g, h)\tau_{gh}$ for all $g, h \in G$. Then $1_{\mathcal{A}} = \tau_1$ and $(\tau_g)^{-1} = c(g, g^{-1})^{-1} t_{g^{-1}}$ for all $g \in G$.

Suppose also that $Z = \langle c(g, h) | g, h \in G\rangle$ is a finite (and hence cyclic) subgroup of $\mathcal{O}^\times$ of order n with $(n, p) = 1$. Let $\hat{G} = Z \times G$ be as above and let $\hat{e} = \frac{1}{n} \sum_{z \in Z} z^{-1}(z, 1) \in \mathcal{O}i(Z) \leq Z(\mathcal{O}\hat{G})$. Then $\hat{e}$ is a block idempotent of $\mathcal{O}i(Z)$ and is central in $\mathcal{O}\hat{G}$. Also $\mathcal{O}\hat{G}$ is a G-crossed product $\mathcal{O}$-algebra with $(\mathcal{O}\hat{G})_g =$

$(\mathcal{O}i(Z))(1,g)$ for all $g \in G$ and $(z,g)\hat{e} = z(1,g)\hat{e}$ for all $z \in Z$ and all $g \in G$. Hence $(\mathcal{O}\hat{G})\hat{e} = \oplus_{g\in G}\mathcal{O}(1,g)\hat{e}$ in $\mathcal{O}$-mod and the $\mathcal{O}$ linear map $\alpha : (\mathcal{O}\hat{G})\hat{e} \to \mathcal{A}$ such that $(1,g)\hat{e} \mapsto \tau_g$ for all $g \in G$ is an $\mathcal{O}$-algebra isomorphism. Here $(\mathcal{O}\hat{G})\hat{e}$ is the direct sum of the block algebras of $\hat{G}$ that cover the block $\hat{e}$ of $\mathcal{O}i(Z)$.

As above, let the finite group G of order m act trivially on the abelian group M. Assume also that M is m-divisible and that $\Omega_m(M) = \{x \in M \mid x^m = 1\}$ is finite.

Let $U = \{c \in Z^2(G,M) \mid c^m = 1\}$, so that U is a finite subgroup of $Z^2(G,M)$. Note that $H^2(G,M)$ has exponent dividing $m = |G|$ ([9, I, Satz 16.19]) and that $B^2(G,M)$ is m-divisible.

The proof of [10, Theorem 11.15] yields:

Lemma 1.4. *Under these conditions, there is a subgroup $W \leq U$ such that $Z^2(G,M) = B^2(G,M) \times W$ and hence $H^2(G,M) \cong W$.*

Let R be a commutative ring.

Lemma 1.5. *Let $\mathcal{A}$ be an R-algebra and let $\mathcal{S}$ be an R-subalgebra of $\mathcal{A}$ such that $\mathcal{S} \tilde{=} M_r(R)$ as R-algebras, where $M_r(R)$ is the R-algebra of all $r \times r$ matrices over R for some positive integer r. Then:*

(a) *$C_{\mathcal{A}}(\mathcal{S}^\times) = C_{\mathcal{A}}(\mathcal{S})$; and*

(b) *$N_{\mathcal{A}^\times}(\mathcal{S}^\times) = N_{\mathcal{A}^\times}(\mathcal{S})$.*

Proof. Clearly $C_{\mathcal{A}}(\mathcal{S}) \subseteq C_{\mathcal{A}}(\mathcal{S}^\times)$ and $N_{\mathcal{A}^\times}(\mathcal{S}) \subseteq N_{\mathcal{A}^\times}(\mathcal{S}^\times)$ since $1_{\mathcal{S}}^\alpha = 1_{\mathcal{S}} \in \mathcal{S}^\times$ for all $\alpha \in N_{\mathcal{A}^\times}(\mathcal{S})$. If $r = 1$, then $\mathcal{S} = R1_{\mathcal{S}}$. Hence $C_{\mathcal{A}}(\mathcal{S}^\times) \subseteq C_{\mathcal{A}}(\mathcal{S})$, $N_{\mathcal{A}^\times}(\mathcal{S}^\times) \subseteq N_{\mathcal{A}^\times}(\mathcal{S})$ and we are done. Assume that $r \geq 2$. Clearly if $x \in \mathcal{S}$, then $x \in \mathcal{S}^\times$ if and only if $\det(x) \in R^\times$. Also $\mathcal{S}$ has an R-basis $\{E_{ij} | 1 \leq i,j \leq r\}$ such that

$$E_{ij}E_{mn} = \begin{cases} 0 & \text{if } j \neq m, \\ E_{in} & \text{if } j = m. \end{cases}$$

Let $\alpha \in C_{\mathcal{A}}(\mathcal{S}^\times)$. Then $\alpha 1_{\mathcal{S}} = 1_{\mathcal{S}}\alpha$ and $\alpha(1_{\mathcal{S}}+E_{ij}) = (1_{\mathcal{S}}+E_{ij})\alpha$ for all $1 \leq i,\ j \leq r$ with $i \neq j$. Thus $\alpha E_{ij} = E_{ij}\alpha$ and $\alpha E_{ii} = \alpha E_{ij}E_{ji} = E_{ij}E_{ji}\alpha = E_{ii}\alpha$ for all $1 \leq i,\ j \leq r$ with $i \neq j$. Consequently (a) holds. A similar argument demonstrates that if $\alpha \in N_{\mathcal{A}^\times}(\mathcal{S}^\times)$, then $1_{\mathcal{S}}^\alpha = 1_{\mathcal{S}}$, $(1_{\mathcal{S}} + E_{ij})^\alpha = 1_{\mathcal{S}} + E_{ij}^\alpha$ and $E_{ij}^\alpha \in \mathcal{S}$ for all $1 \leq i,j \leq r$ with $i \neq j$. Thus $E_{ii}^\alpha = (E_{ij}E_{ji})^\alpha = E_{ij}^\alpha E_{ji}^\alpha \in \mathcal{S}$ for all $1 \leq i,j \leq r$ with $i \neq j$, $\mathcal{S}^\alpha \subseteq \mathcal{S}$ and $\mathcal{S}^{\alpha^{-1}} \subseteq \mathcal{S}$ and hence $\mathcal{S}^\alpha = \mathcal{S}$ and (b) follows.

Let $\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ be a G-graded R-algebra (cf. [4]). For each $X \subseteq G$, let $\mathcal{A}_X = \oplus_{x\in X}\mathcal{A}_x$, so that if $H \leq G$, then $\mathcal{A}_H$ is an H-graded R-subalgebra of $\mathcal{A}$ since $1_{\mathcal{A}} \in \mathcal{A}_1$.

Lemma 1.6. *Let $H \leq G$, let b be an idempotent of $\mathcal{A}_H$ and let $\alpha \in b\mathcal{A}_H b$. Suppose that there is an element $\beta \in \mathcal{A}$ such that $\alpha\beta = b$ and $\beta\alpha = b$ (so that $\alpha(b\beta b) = b$ and $(b\beta b)\alpha = b$). Then $b\beta b \in b\mathcal{A}_H b$.*

Proof. First suppose that $b = 1_{\mathcal{A}}$ and let $\mathcal{T}$ be a right transversal of H in G with $1 \in \mathcal{T}$ so that $G = \bigcup_{t\in\mathcal{T}} Ht$, where the union is disjoint. Let $\beta = \sum_{\substack{h\in H \\ t\in\mathcal{T}}} b_{ht}$, where $b_{ht} \in \mathcal{A}_{ht}$ for all $h \in H$ and $t \in \mathcal{T}$. Then $1_{\mathcal{A}} = \sum_{t\in\mathcal{T}}(\alpha(\sum_{h\in H} b_{ht})) \in \mathcal{A}_1$, where $\alpha(\sum_{h\in H} b_{ht}) \in \mathcal{A}_{Ht}$ for all $t \in \mathcal{T}$. Thus if $t \neq 1$, then $\alpha(\sum_{h\in H} b_{ht}) = 0$ and so $(\sum_{h\in H} b_{ht}) = 0$ since α is a unit. Consequently $\beta \in \mathcal{A}_H$. For the general case, set $\alpha^* = \alpha + (1_{\mathcal{A}} - b)$ and $\beta^* = b\beta b + (1_{\mathcal{A}} - b)$. Then $\alpha^*\beta^* = 1_{\mathcal{A}} = \beta^*\alpha^*$, where $\alpha^* \in \mathcal{A}_H$. By the case above, we conclude that $\beta^* = b\beta b + (1_{\mathcal{A}} - b) \in \mathcal{A}_H$ and hence $b\beta b \in b\mathcal{A}_H b$ and we are done.

Let $\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ be a G-graded crossed-product R-algebra (with an identity $1_{\mathcal{A}} \in \mathcal{A}_1$) and assume that $\mathcal{A}_1$ is commutative. Thus $\mathcal{A}_g \cap \mathcal{A}^\times$ is nonempty for each $g \in G$. Here we have the short exact sequence of groups

$$1 \to \mathcal{A}_1^\times \to \mathcal{G}r(\mathcal{A}^\times) \stackrel{\deg}{\to} G \to 1$$

and G acts on $\mathcal{A}_1$ (on the right) as defined by

$$\alpha^g = \alpha^{u_g} = u_g^{-1}\alpha u_g \text{ for all } \alpha \in \mathcal{A}_1$$

for some (and hence every) $u_g \in \mathcal{A}_g \cap \mathcal{A}^\times$ for each $g \in G$.

For each $g \in G$, choose $u_g \in \mathcal{A}_g \cap \mathcal{A}^\times$. Here $\mathcal{A}_g = \mathcal{A}_1 u_g = u_g\mathcal{A}_1$ and $u_g u_h = u_{gh}c(g,h)$ for a unique $c(g,h) \in \mathcal{A}_1^\times$ for all $g, h \in G$ and $c \in Z^2(G, \mathcal{A}_1^\times)$. Clearly multiplication in $\mathcal{A}$ is determined by $\mathcal{A}_1$, the action of G on $\mathcal{A}_1$ and the element $c \in Z^2(G, \mathcal{A}_1^\times)$. Also, the element $cB^2(G, \mathcal{A}_1^\times)$ is independent of the choices of the elements $u_g \in \mathcal{A}_g \cap \mathcal{A}^\times$ for all $g \in G$.

Let $\mathcal{B} = \oplus_{g\in G}\mathcal{B}_g$ also be a G-graded crossed product R-algebra (with an identity $1_{\mathcal{B}} \in \mathcal{B}_1$) and let $\sigma : \mathcal{A}_1 \to \mathcal{B}_1$ be an R-algebra isomorphism (so that $\mathcal{B}_1$ is commutative) such that σ commutes with the action of G on $\mathcal{A}_1$ and on $\mathcal{B}_1$ (i.e., $\sigma(\alpha^g) = \sigma(\alpha)^g$ for all $\alpha \in \mathcal{A}_1$ and all $g \in G$). Let $s : H^2(G, \mathcal{A}_1^\times) \to H^2(G, \mathcal{B}_1^\times)$ be the group isomorphism induced by σ.

The following result is easy to prove and well known:

Proposition 1.7. *There is a G-graded R-algebra isomorphism $\tilde{\sigma} : \mathcal{A} \to \mathcal{B}$ that extends σ if and only if $s(cB^2(G, \mathcal{A}_1^\times))$ is the element of $H^2(G, \mathcal{B}_1^\times)$ determined by the action of G on $\mathcal{B}_1$ via $\mathcal{B}$.*

As in the Introduction, we let $\mathcal{O}$ be a complete discrete valuation ring of characteristic zero such that $k = \mathcal{O}/J(\mathcal{O})$ is an algebraically closed field of prime characteristic p. Let $\mathcal{K}$ denote the field of fractions of $\mathcal{O}$. Such rings exist (cf. [17, II, Theorem 3]) and satisfy [18, assumption (2.1)]. Moreover by [17, II, Proposition 8]:

(1.2) there is a multiplicative injection $f : k \to \mathcal{O}$ such that

(a) $f(y) + J(\mathcal{O}) = y$ for all $y \in k = \mathcal{O}/J(\mathcal{O})$; and

(b) if $x \in \mathcal{O}$, then $x \in f(k)$ if and only if x is a p^n-power in $\mathcal{O}$ for every integer $n \geq 0$.

Thus $f(1_k) = 1_{\mathcal{O}}$, $f(k^\times) \leq \mathcal{O}^\times$, $\mathcal{O}^\times = f(k^\times) \times (1 + J(\mathcal{O}))$, any element of $\mathcal{O}^\times$ of finite order prime to p lies in $f(k^\times)$ (cf. [18, Lemma 2.3]) and $f : k^\times \to f(k^\times)$ is a group isomorphism.

For the remainder of this section, we shall also assume that every $\mathcal{O}$-algebra $\mathcal{A}$ is finitely generated in $\mathcal{O}$-mod. Consequently $\mathcal{A}/J(\mathcal{A})$ is a finite-dimensional split semi-simple k-algebra. We denote the unity element of $\mathcal{A}$ by $1_{\mathcal{A}}$ (or sometimes simply by 1). Also we shall assume that every $\mathcal{A}$-module is a finitely generated and unitary $\mathcal{A}$-module. Note that every free $\mathcal{O}$-module is a torsion free $\mathcal{O}$-module.

If r is a positive integer, then $M_r(k)$ and $M_r(\mathcal{O})$ will denote the $\mathcal{O}$-algebras of all $r \times r$ matrices over k and $\mathcal{O}$, respectively. As in [18, Section 7], an $\mathcal{O}$-algebra $\mathcal{S}$ is called $\mathcal{O}$-simple if $\mathcal{S}$ is $\mathcal{O}$-algebra isomorphic to some $M_r(\mathcal{O})$ and $\mathcal{S}$ is said to be $\mathcal{O}$-semi-simple if $\mathcal{S}$ is $\mathcal{O}$-algebra isomorphic to a direct sum of $\mathcal{O}$-simple $\mathcal{O}$-algebras.

Let $\mathcal{A}$ be an $\mathcal{O}$-algebra and let m be a positive integer with $(m, p) = 1$. Let $\mu : \mathcal{A} \to \mathcal{A}$ be the function such that $x \mapsto x^m$ for all $x \in \mathcal{A}$.

Proposition 1.8. *μ induces a bijection of $1_{\mathcal{A}} + J(\mathcal{A})$ onto itself.*

To prove this, we set $J = J(\mathcal{A})$ and note the following two trivialities.

Lemma 1.9. *Let $\mathcal{G}$ be a not necessarily finite group and let $x, y \in \mathcal{G}$ be such that $x^m = y$ and $|x| \mid p^k$ for some positive integer $k \geq 1$. Then $x = y^a$ for any $a, b \in Z$ such that $1 = ma + p^k b$. Hence if x_1, x_2 are p-elements of $\mathcal{G}$ such that $x_1^m = y = x_2^m$, then $x_1 = x_2$.*

Lemma 1.10. *Let n, k be positive integers. Then:*

(a) *$(J^n/J^{n+1}, +)$ is an abelian group of exponent p;*

(b) *the map $J^n \mapsto 1_{\mathcal{A}} + J^n$ such that $a \mapsto 1_{\mathcal{A}} + a$ for all $a \in J^n$ induces a group isomorphism of $(J^n/J^{n+1}, +)$ onto the multiplicative group $(1_{\mathcal{A}} + J^n)/(1_{\mathcal{A}} + J^{n+1})$; and*

(c) *the multiplicative group $(1_{\mathcal{A}} + J^n)/(1_{\mathcal{A}} + J^{n+k})$ has exponent dividing p^k.*

Proof of Proposition 1.8. By [18, Lemma 45.5], we have

$$1_{\mathcal{A}} + J = \varprojlim((1_{\mathcal{A}} + J)/(1_{\mathcal{A}} + J^n)).$$

For each integer $k \geq 1$, choose $a_k, b_k \in Z$ such that $ma_k + p^k b_k = 1$. Assume that $a_0 = 0$ and $b_0 = 1$. Let $y \in 1_{\mathcal{A}} + J$. If $k \geq 1$, then $y^{a_k}(1 + J^{k+1})$ is the unique element w of $(1_{\mathcal{A}} + J)/(1_{\mathcal{A}} + J^{k+1})$ such that $w^m = y(1_{\mathcal{A}} + J^{k+1})$ by Lemmas 1.9 and 1.10(c). Moreover, $(y^{a_k}(1_{\mathcal{A}} + J^k))^m = y(1_{\mathcal{A}} + J^k)$ so that $y^{a_k}(1_{\mathcal{A}} + J^k) = y^{a_{k-1}}(1_{\mathcal{A}} + J^k)$. Thus $u = (y(1_{\mathcal{A}} + J),\ y^{a_1}(1_{\mathcal{A}} + J^2),\ y^{a_2}(1_{\mathcal{A}} + J^3), \dots)$ is an element of $\varprojlim((1_{\mathcal{A}} + J)/(1_{\mathcal{A}} + J^n))$ such that

$$u^m = (y(1_{\mathcal{A}} + J), y(1_{\mathcal{A}} + J^2),\ y(1_{\mathcal{A}} + J^3), \dots) = y.$$

Assume that $z = (z_0(1_{\mathcal{A}} + J),\ z_1(1_{\mathcal{A}} + J^2),\ z_2(1_{\mathcal{A}} + J^3), \dots)$ is an element of $\varprojlim((1_{\mathcal{A}} + J)/(1_{\mathcal{A}} + J^n))$ such that $z^m = y$. Then $(z_k(1_{\mathcal{A}} + J^{k+1}))^m = y(1_{\mathcal{A}} + J^{k+1})$ and Lemmas 1.9 and 1.10(c) imply that

$$z_k(1_{\mathcal{A}} + J^{k+1}) = y^{a_k}(1_{\mathcal{A}} + J^{k+1})$$

for all $k \geq 0$. Thus $z = u$ and we are done.

We continue with this situation. We shall require the following extension of [18, Lemma 45.6].

Let X be a group containing $1_{\mathcal{A}} + J(\mathcal{A})$ as a normal subgroup. Assume also that the subgroup $1_{\mathcal{A}} + (J(\mathcal{A})^n)$ is normal in X for every integer $n \geq 1$ and that E is a finite subgroup of X with $|E| = m$ relatively prime to p such that $X = (1_{\mathcal{A}} + J(\mathcal{A}))E$. Thus $E \cap (1_{\mathcal{A}} + J(\mathcal{A})) = 1_{\mathcal{A}}$ by Proposition 1.8.

Lemma 1.11. *Let $e \in E$ and let $x \in (1_{\mathcal{A}} + J(\mathcal{A}))e$ be of order prime to p. Then there is an element $j \in J(\mathcal{A})$ such that $x^{1_{\mathcal{A}}+j} = e$.*

Proof. Clearly we may assume that $X = (1_{\mathcal{A}} + J(\mathcal{A}))\langle e\rangle$. Let $\alpha = |x|$ and $\beta = |e|$. Then $x^\alpha \in 1_{\mathcal{A}} + J(\mathcal{A})$ and hence $\alpha = \beta s$ for some positive integer s. Here $x^\beta \in 1_{\mathcal{A}} + J(\mathcal{A})$ and $(x^\beta)^s = 1_{\mathcal{A}}$. Since $(\alpha, p) = 1$, we conclude that $(s, p) = 1$. Thus $x^\beta = 1_{\mathcal{A}}$ because of Proposition 1.8. Consequently $\alpha = \beta$. Since $X = (1_{\mathcal{A}} + J(\mathcal{A}))\langle x\rangle$ and $\langle x\rangle \cap (1_{\mathcal{A}} + J(\mathcal{A})) = 1_{\mathcal{A}}$, [18, Lemma 45.6] yields an element $j \in J(\mathcal{A})$ such that $\langle x\rangle^{1_{\mathcal{A}}+j} = \langle e\rangle$. Then $x^{1_{\mathcal{A}}+j} \in \langle e\rangle \cap ((1 + J(\mathcal{A}))e) = \{e\}$ and we are done.

Let $\mathcal{A}$ be an $\mathcal{O}$-free $\mathcal{O}$-algebra. Then, as in [18, Theorem 7.3], there is an $\mathcal{O}$-semi-simple $\mathcal{O}$-subalgebra $\mathcal{S}$ such that $\mathcal{A} = \mathcal{S} + J(\mathcal{A})$ and, in fact, $\mathcal{S}$ is a maximal $\mathcal{O}$-semi-simple $\mathcal{O}$-subalgebra of $\mathcal{A}$. We require the following observations in this context.

Proposition 1.12. (a) *$J(\mathcal{A}) \cap \mathcal{S} = J(\mathcal{O})\mathcal{S} = J(\mathcal{S})$ and $\mathcal{S}/(J(\mathcal{O})\mathcal{S})$ is a k-semi-simple k-algebra;*

(b) *$1_{\mathcal{A}} = 1_{\mathcal{S}}$ and $\mathcal{S} \cap (1_{\mathcal{A}} + J(\mathcal{A})) = \mathcal{S}^{\times} \cap (1_{\mathcal{A}} + J(\mathcal{A})) = 1_{\mathcal{A}} + J(\mathcal{S})$;*

(c) *$\mathcal{A}^{\times} = (1_{\mathcal{A}} + J(\mathcal{A}))\mathcal{S}^{\times} = \mathcal{S}^{\times}(1_{\mathcal{A}} + J(\mathcal{A}))$; and*

(d) *any two maximal $\mathcal{O}$-semi-simple $\mathcal{O}$-subalgebras of $\mathcal{A}$ are conjugate by an element of $1_{\mathcal{A}} + J(\mathcal{A})$.*

(e) *Assume also that $\mathcal{S}$ is $\mathcal{O}$-simple. Then $C_{\mathcal{A}}(\mathcal{S}) = C_{\mathcal{A}}(\mathcal{S}^{\times})$, $N_{\mathcal{A}^{\times}}(\mathcal{S}^{\times}) = N_{\mathcal{A}^{\times}}(\mathcal{S}) = \mathcal{S}^{\times} N_{1_{\mathcal{A}}+J(\mathcal{A})}(\mathcal{S}) = N_{1_{\mathcal{A}}+J(\mathcal{A})}(\mathcal{S})\mathcal{S}^{\times}$ and $[\mathcal{S}^{\times}, N_{1_{\mathcal{A}}+J(\mathcal{A})}(\mathcal{S}^{\times})] \leq \mathcal{S}^{\times} \cap (1_{\mathcal{A}} + J(\mathcal{A})) = 1_{\mathcal{A}} + J(\mathcal{S})$.*

Proof. Clearly (a) holds and, since $1_{\mathcal{A}} - 1_{\mathcal{S}}$ is an idempotent in $J(\mathcal{A})$, (b) also follows. Clearly $(1_{\mathcal{A}} + J(\mathcal{A}))\mathcal{S}^{\times} \leq \mathcal{A}^{\times}$. Let $s \in \mathcal{S}$ and $j \in J(\mathcal{A})$ be such that $s + j \in \mathcal{A}^{\times}$. Then there is an element $t \in \mathcal{S}$ such that $st - 1_{\mathcal{A}} \in J(\mathcal{A}) \cap \mathcal{S} = J(\mathcal{S})$ and $ts - 1_{\mathcal{A}} \in J(\mathcal{A}) \cap \mathcal{S} = J(\mathcal{S})$. Thus $st \in 1_{\mathcal{A}} + J(\mathcal{S})$ and $ts \in 1_{\mathcal{A}} + J(\mathcal{S})$ so that s has an inverse s^{-1} in $\mathcal{S}$. Then $s + j = s(1_{\mathcal{A}} + s^{-1}j) \in \mathcal{S}^{\times}(1 + J(\mathcal{A}))$, (c) holds and (d) follows from [18, Theorem 7.3(c)]. Finally Lemma 1.5, (b) and (c) and the fact that $1 + J(\mathcal{A}) \leq \mathcal{A}^{\times}$ yield (e).

Note that we always assume that any finite group G acts trivially on $\mathcal{O}$.

Lemma 1.13. *Let $\mathcal{A}$ be an $\mathcal{O}$-free G-algebra such that $\mathcal{A} = \mathcal{O}1_{\mathcal{A}} + J(\mathcal{A})$. Then:*

(a) *$\mathcal{A}^{\times} = f(k^{\times})1_{\mathcal{A}} \times (1_{\mathcal{A}} + J(\mathcal{A}))$ as a G-group (since G acts trivially on $\mathcal{O}1_{\mathcal{A}}$ by hypothesis); and*

(b) *if $|G| = m$ is prime to p and $\mathcal{A}$ is also commutative, then $f : k^{\times} \to \mathcal{O}^{\times}$ of (1.2) induces a natural group isomorphism $f^* : k^{\times} \to f(k^{\times})1_{\mathcal{A}}$ such that $x \mapsto f(x)1_{\mathcal{A}}$ for all $x \in k^{\times}$ and f^* induces a group isomorphism $H^2(G, f^*) : H^2(G, k^{\times}) \to H^2(G, \mathcal{A}^{\times})$.*

Proof. Since $\mathcal{O}^{\times}1_{\mathcal{A}} = f(k^{\times})1_{\mathcal{A}} \times (1 + J(\mathcal{O}))1_{\mathcal{A}}$ and $\mathcal{A}^{\times} = (\mathcal{O}^{\times}1_{\mathcal{A}})(1_{\mathcal{A}} + J(\mathcal{A}))$, we have $\mathcal{A}^{\times} = (f(k^{\times})1_{\mathcal{A}})(1_{\mathcal{A}} + J(\mathcal{A}))$. Let $x \in (f(k^{\times})1_{\mathcal{A}}) \cap (1_{\mathcal{A}} + J(\mathcal{A}))$. Then $x = \alpha 1_{\mathcal{A}} = 1_{\mathcal{A}} + j$, where $\alpha \in f(k^{\times})$ and $j \in J(\mathcal{A})$. Thus $(\alpha - 1)1_{\mathcal{A}} \in J(\mathcal{A}) \cap (\mathcal{O}1_{\mathcal{A}}) = J(\mathcal{O})1_{\mathcal{A}}$, $\alpha = 1$ and $x = 1_{\mathcal{A}}$ which proves (a). In the situation of (b), we have $H^2(G, \mathcal{A}^{\times}) = H^2(G, f(k^{\times})1_{\mathcal{A}}) \times H^2(G, 1_{\mathcal{A}} + J(\mathcal{A}))$ since the direct decomposition of (a) is a G-decomposition. As $m = |G|$ is relatively prime to p and the m-th power map is an automorphism of $1_{\mathcal{A}} + J(\mathcal{A})$ by Proposition 1.8, we have

$$H^2(G, 1_{\mathcal{A}} + J(\mathcal{A})) = 1$$

(by [9, I, Satz 16.19(a)]). Our proof is complete.

Let D be a finite p-group and set $\mathcal{A} = \mathcal{O}D$. Let $\mathcal{I}(\mathcal{O}D) = \sum_{d \in D^{\#}} \mathcal{O}(d-1)$ be the augmentation ideal of $\mathcal{A}$, so that $\mathcal{I}(\mathcal{O}D) \leq J(\mathcal{A})$, $\mathcal{A} = \mathcal{O}1_{\mathcal{A}} \oplus \mathcal{I}(\mathcal{O}D)$ in $\mathcal{O}$-mod and $\mathcal{O}/\mathcal{I}(\mathcal{O}D) \tilde{=} \mathcal{O}$ as $\mathcal{O}$-algebras.

Lemma 1.14. (a)

$$\mathcal{A}^{\times} = (\mathcal{O}^{\times}1_{\mathcal{A}}) \times (1_{\mathcal{A}} + \mathcal{I}(\mathcal{O}D))$$

and

$$1_{\mathcal{A}} + J(\mathcal{O}D) = (1_{\mathcal{A}} + J(\mathcal{O})1_{\mathcal{A}}) \times (1_{\mathcal{A}} + \mathcal{I}(\mathcal{O}D))$$

as groups and $(\mathcal{O}^{\times}1_{\mathcal{A}}) \cap (1_{\mathcal{A}} + J(\mathcal{O}D)) = 1_{\mathcal{A}} + J(\mathcal{O})1_{\mathcal{A}}$; and

(b) *let m be a positive integer with $(m, p) = 1$ and let $\mu : 1_{\mathcal{A}} + J(\mathcal{A}) \to 1_{\mathcal{A}} + J(\mathcal{A})$ be the m-th power bijective map of Proposition 1.8. Then the restriction of μ to $1_{\mathcal{A}} + \mathcal{I}(\mathcal{O}D) \to 1_{\mathcal{A}} + \mathcal{I}(\mathcal{O}D)$ is a bijection.*

Proof. Let $x \in \mathcal{A}^\times$. Then $x = a1_\mathcal{A} + i$, where $i \in \mathcal{I}(\mathcal{O}D)$ and $a \in \mathcal{O}$. Since $\mathcal{A}/\mathcal{I}(\mathcal{O}D) \tilde{=} \mathcal{O}$, $a \in \mathcal{O}^\times$ and $x = (a1_\mathcal{A})(1_\mathcal{A} + a^{-1}i) \in (\mathcal{O}^\times 1_\mathcal{A})(1_\mathcal{A} + \mathcal{I}(\mathcal{O}D))$. Let $b \in \mathcal{O}^\times$ be such that $b1_\mathcal{A} = 1_\mathcal{A} + i$, where $i \in \mathcal{I}(\mathcal{O}D)$. Then $(b-1)1_\mathcal{A} = i = 0$, $b = 1$ and $\mathcal{A}^\times = (\mathcal{O}^\times 1_\mathcal{A}) \times (1_\mathcal{A} + \mathcal{I}(\mathcal{O}D))$ as groups. Since $1_\mathcal{A} + \mathcal{I}(\mathcal{O}D) \leq 1_\mathcal{A} + J(\mathcal{O}D)$, $1_\mathcal{A} + J(\mathcal{O}D) = ((1 + J(\mathcal{O}D)) \cap (\mathcal{O}^\times 1_\mathcal{A})) \times (1_\mathcal{A} + \mathcal{I}(\mathcal{O}D))$. Let $a \in \mathcal{O}^\times$ be such that $a1_\mathcal{A} = 1_\mathcal{A} + j$, where $j \in J(\mathcal{O}D)$. Then $(a-1)1_\mathcal{A} = j \in J(\mathcal{O}D) \cap (\mathcal{O}1_\mathcal{A}) = J(\mathcal{O})1_\mathcal{A}$ and (a) holds.

For (b), it suffices to prove that $\mu : 1_\mathcal{A} + \mathcal{I}(\mathcal{O}D) \to 1_\mathcal{A} + \mathcal{I}(\mathcal{O}D)$ is surjective. Let $i \in \mathcal{I}(\mathcal{O}D)$. Then $1_\mathcal{A} + i = (1_\mathcal{A} + j)^m$ for some $j \in J(\mathcal{O}D)$ by Proposition 1.8. Also $1_\mathcal{A} + j = a1_\mathcal{A} + s$ for some $a \in \mathcal{O}$ and some $s \in \mathcal{I}(\mathcal{O}D)$. As $\mathcal{A}/\mathcal{I}(\mathcal{O}D) \tilde{=} \mathcal{O}$ as rings, $a^m = 1$. Thus $1_\mathcal{A} + j = a1_\mathcal{A}(1_\mathcal{A} + a^{-1}s)$, where $1_\mathcal{A} + a^{-1}s \in 1_\mathcal{A} + J(\mathcal{A})$. Now Lemma 1.13(a) implies that $a = 1$, $j = s$ and we are done.

Next assume that $\mathcal{A}$ is a G-graded crossed-product $\mathcal{O}$-algebra that is $\mathcal{O}$-free. Note that $1_\mathcal{A} \in \mathcal{A}_1$.

Proposition 1.15. *Assume that $\mathcal{A}_1/J(\mathcal{A}_1) \cong M_r(k)$ for some positive integer r. Let $\mathcal{S}$ be a maximal $\mathcal{O}$-semisimple subalgebra of $\mathcal{A}_1$ so that $1_\mathcal{A} \in \mathcal{S}$ and let $u_g \in \mathcal{A}_g \cap \mathcal{A}^\times$ for each $g \in G$. Then:*

(a) *$\mathcal{S} \cong M_r(\mathcal{O})$ as $\mathcal{O}$-algebras, $\mathcal{A}_1 = \mathcal{S} + J(\mathcal{A}_1)$, $\mathcal{S} \cap J(\mathcal{A}_1) = J(\mathcal{O})\mathcal{S} = J(\mathcal{S})$ and $\mathcal{A}_1^\times = \mathcal{S}^\times(1_\mathcal{A} + J(\mathcal{A}_1)) = (1_\mathcal{A} + J(\mathcal{A}_1))\mathcal{S}^\times$;*

(b) *for each $g \in G$, there is an element $w_g \in \mathcal{A}_1^\times$ such that $v_g = w_g u_g \in C_{\mathcal{A}_g}(\mathcal{S}) \cap \mathcal{A}^\times$;*

(c) *$C_\mathcal{A}(\mathcal{S}) = \oplus_{g\in G}(C_{\mathcal{A}_1}(\mathcal{S})v_g)$ is a G-graded crossed-product $\mathcal{O}$-subalgebra of $\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$, where $(C_\mathcal{A}(\mathcal{S}))_g = C_{\mathcal{A}_1}(\mathcal{S})v_g$ for all $g \in G$;*

(d) *multiplication $\mu : \mathcal{S} \otimes_\mathcal{O} C_\mathcal{A}(\mathcal{S}) \to \mathcal{A}$ such that $\sigma \otimes_\mathcal{O} \rho \mapsto \sigma\rho = \rho\sigma$ for all $\sigma \in \mathcal{S}$ and all $\rho \in C_\mathcal{A}(\mathcal{S})$ is a G-graded $\mathcal{O}$-algebra isomorphism (where $(\mathcal{S} \otimes_\mathcal{O} C_\mathcal{A}(\mathcal{S}))_g = \mathcal{S} \otimes_\mathcal{O} (C_\mathcal{A}(\mathcal{S})_g)$ for all $g \in G$); and*

(e) *$\mathcal{A}J(\mathcal{A}_1)$ is a G-graded ideal of $\mathcal{A}$ and, setting $\tilde{\mathcal{A}} = \mathcal{A}/(\mathcal{A}J(\mathcal{A}_1))$, $\tilde{\mathcal{A}}$ becomes a G-graded crossed-product k-algebra with $\tilde{\mathcal{A}}_g = \mathcal{A}_g + \mathcal{A}J(\mathcal{A}_1)$ for all $g \in G$ and $\tilde{\mathcal{A}}_1 = \tilde{\mathcal{S}} \cong M_r(k)$ as k-algebras. Moreover, $C_{\tilde{\mathcal{A}}_1}(\tilde{\mathcal{A}}_1) = Z(\tilde{\mathcal{A}}_1) = \widetilde{\mathcal{O}1_\mathcal{A}} \cong k$, $C_{\tilde{\mathcal{A}}}(\tilde{\mathcal{A}}_1) = \oplus_{g\in G}C_{\tilde{\mathcal{A}}_1}(\tilde{\mathcal{A}}_1)\tilde{v}_g$ is a G-graded twisted group k-subalgebra of $\tilde{\mathcal{A}} = \oplus_{g\in G}\tilde{\mathcal{A}}_g$ and multiplication $\tilde{\mu} : \tilde{\mathcal{A}}_1 \otimes_k C_{\tilde{\mathcal{A}}}(\tilde{\mathcal{A}}_1) \to \tilde{\mathcal{A}}$ such that $\tilde{\sigma} \otimes_k \tilde{\rho} \mapsto \tilde{\sigma}\tilde{\rho} = \tilde{\rho}\tilde{\sigma}$ for all $\tilde{\sigma} \in \tilde{\mathcal{A}}_1$ and all $\tilde{\rho} \in C_{\tilde{\mathcal{A}}}(\tilde{\mathcal{A}}_1)$ is a G-graded k-algebra isomorphism (where $(\tilde{\mathcal{A}}_1 \otimes_k C_{\tilde{\mathcal{A}}}(\tilde{\mathcal{A}}_1))_g = \tilde{\mathcal{A}}_1 \otimes_k (C_{\tilde{\mathcal{A}}}(\tilde{\mathcal{A}}_1)_g)$ for all $g \in G$).*

Remark 1.16. In Proposition 1.15(e), $C_{\tilde{\mathcal{A}}}(\tilde{\mathcal{A}}_1) = \oplus_{g\in G}C_{\tilde{\mathcal{A}}_1}(\tilde{\mathcal{A}}_1)\tilde{v}_g$ is the "Clifford extension" of [3, (1.10) and (4.2)].

Proof. By [18, Theorem 7.3] and Proposition 1.12, we have (a). Fix $g \in G$. Since ${}^{u_g}\mathcal{S}$ is also a maximal $\mathcal{O}$-semisimple subalgebra of $\mathcal{A}_1$, there is an element $j \in J(\mathcal{A}_1)$ such that ${}^{(1+j)u_g}\mathcal{S} = \mathcal{S}$. As $\mathrm{Aut}_\mathcal{O}(\mathcal{S}) = \mathrm{Inn}_\mathcal{O}(\mathcal{S})$ (cf. [18, Theorem 7.2]), there is an element $s_g \in \mathcal{S}^\times$ such that $s_g(1 + j)u_g \in C_{\mathcal{A}_g}(\mathcal{S}) \cap \mathcal{A}^\times$ and we may take $w_g = s_g(1 + j) \in \mathcal{A}_1^\times$ to conclude (b). Then (c) is immediate and (d) follows from [18, Proposition 7.5]. For similar reasons (e) also holds.

In the remainder of this section, G will, as usual, denote a finite group and we shall also assume that $(\mathcal{K}, \mathcal{O}, k)$ is "big enough" for all subgroups of G. Let b be a block idempotent of $\mathcal{O}G$ and let $- : \mathcal{O}G \to kG$ be the $\mathcal{O}$-algebra epimorphism induced by the natural epimorphism $- : \mathcal{O} \to k = \mathcal{O}/J(\mathcal{O})$. Thus $\bar{b}$ is a block idempotent of kG.

We shall need the following well-known "facts":

Lemma 1.17. (a) *If D is a defect group of b, then D is a defect group of $\bar{b}$; and* (b) *if P is a p-subgroup of G, then the $\mathcal{O}P$-module homorphism $m(b) : \mathcal{O}P \to (\mathcal{O}G)b$ such that $p \mapsto pb$ for all $p \in P$ is an injection and the kP-module homorphism $\overline{m(b)} : kP \to (kG)\bar{b}$ induced by $m(b)$ is an injection.*

Proof. Here (a) is a consequence of [5, III, Theorem 6.10]. Since $(\mathcal{O}G)b$ is projective and hence free in $\mathcal{O}P$-mod, the first part of (b) holds and similar arguments yield the remainder of (b).

Next assume that b is a block of defect 0 of the finite group G and let V be an indecomposable module in $(\mathcal{O}G)b$-mod. Then V is projective in $\mathcal{O}Gb$-mod, $\bar{V}$ is irreducible in kGb-mod and $\mathcal{K} \otimes_{\mathcal{O}} V$ is irreducible in $\mathcal{K}Gb$-mod. Let χ denote the character of $\mathcal{K} \otimes_{\mathcal{O}} V$ and let φ denote the Brauer character of $\bar{V}$. Let $H \leq G$ and let β be a block of defect 0 of H and let W be an indecomposable module in $(\mathcal{O}H)\beta$-mod so that we have similar conditions as above. Also let δ denote the character of $\mathcal{K} \otimes_{\mathcal{O}} W$ and let ψ denote the Brauer character of $\bar{W}$. Finally let ω denote the multiplicity of $\bar{W}$ as a composition factor of $\mathrm{Res}^G_H(\bar{V})$.

Lemma 1.18. $\omega = (\mathrm{Res}^G_H(\chi), \delta)_H$.

Proof. Clearly $\omega = \frac{1}{|H|}\sum_{h \in H_{p'}} \varphi(h)\psi(h^{-1})$. Since χ and δ vanish on p-singular elements and χ and φ and δ and ψ agree on p-regular elements, we are done.

In the remainder of this section let the solvable finite group $\mathcal{A}$ act on G on the right and be such that $(|G|, |A|) = 1$ so that we are in the Glauberman correspondence situation (cf. [10, Chapter 13]). Set $C = C_G(A)$.

Let $N \trianglelefteq G$ be A-invariant. Then $C_N(A) = C \cap N \trianglelefteq C$ and $\pi(N, A) : \mathrm{Irr}(N)^A \to \mathrm{Irr}(C \cap N)$ is a bijection.

Lemma 1.19. *Let $\chi \in \mathrm{Irr}(N)^A$ and let $c \in C$. Then:*
(a) $\chi^c \in \mathrm{Irr}(N)^A$*;*
(b) $(\pi(N, A)(\chi))^c = \pi(N, A)(\chi^c)$*; and*
(c) $\mathrm{Stab}_C(\chi) = \mathrm{Stab}_C(\pi(N, A)(\chi))$.

Proof. Clearly (a) holds and, for (b), [10, Theorem 13.1] implies that we may assume that A is a q-group for some prime $q \neq p$. Then $\pi(N, A)(\chi)$ is the unique irreducible constituent of $\mathrm{Res}^N_{C\cap N}(\chi)$ with multiplicity prime to q by [10, Theorem 13.4]. Thus $\pi(N, A)(\chi)^c$ is the unique irreducible constituent of $\mathrm{Res}^N_{C\cap N}(\chi^c)$ with multiplicity prime to q. Thus (b) follows and (c) is immediate.

Let D be a p-subgroup of $C \cap N$ and let $\beta \in B\ell(N)^A$ have D as a defect group. Then $W(\beta)$ is a block of $C \cap N$ with defect group D and $\mathrm{Irr}(W(\beta)) = \{\pi(N, A)(\chi) \mid \chi \in \mathrm{Irr}(\beta)\}$ (where $\mathrm{Irr}(\beta) \subseteq \mathrm{Irr}(N)^A$).

Our next result is an immediate consequence of Lemma 1.19.

Corollary 1.20. *Let $c \in C$ (with β and D as above). Then:*
(a) $\beta^c \in B\ell(N)^A$ *and has D^c as a defect group;*
(b) $W(\beta^c) = W(\beta)^c$*; and*
(c) $\mathrm{Stab}_C(\beta) = \mathrm{Stab}_C(W(\beta))$.

We continue with the hypotheses above.

Lemma 1.21. (a) *Let $\beta \in B\ell(N)^A$ have D as a defect group. Then there is an A-invariant block b of G that covers β and such that $D = N \cap \Delta$ for some defect group Δ of b;*

(b) *assume that $D \trianglelefteq N$ and let b be an A-invariant block of G with a defect group D. Then $D = O_p(G) = O_p(N)$ and there is an A-invariant block β of N covered by b such that D is a defect group of β; and*

(c) *let $b \in B\ell(G)^A$ have D as a defect group and cover $\beta \in B\ell(N)^A$ such that D is a defect group of β. Then $W(b) \in B\ell(C)$ (with defect group D) covers $W(\beta)$ of $C \cap N$ (with defect group D).*

Proof. Assume the conditions of (a) and let $\psi \in \mathrm{Irr}(\beta)$, so that ψ is A-stable by [19, Proposition 1]. Then [10, Theorem 13.28] yields an A-stable irreducible constituent χ of $\mathrm{Ind}_N^G(\psi)$ which must belong to an A-stable block b of G that covers β. Thus [12, Proposition 4.2], completes the proof.

Assume the conditions of (b). Thus $D \leq O_p(G)$ and hence $D = O_p(G)$ since D is a defect group of b. Let $\chi \in \mathrm{Irr}(b)$ so that χ is A-stable. Here [10, Theorem 13.27] yields an A-stable irreducible constituent ψ of $\mathrm{Res}_N^G(\chi)$. Then ψ lies in an A-stable block β of N. Let Δ be a defect group of β, so that $D = O_p(N) \leq \Delta$. As above $\Delta \leq \Delta'$ for some defect group Δ' of b. Since $|D| = |\Delta'|$, we conclude that $D = \Delta = \Delta'$ is a defect group of β.

Assume the conditions of (c), let $\chi \in \mathrm{Irr}(b)$ and let $\psi \in \mathrm{Irr}(\beta)$ be such that $[\mathrm{Res}_N^G(\chi), \psi]_N \neq 0$. Thus $[\chi, \mathrm{Ind}_N^G(\psi)]_G \neq 0$ and [10, Theorem 13.29] implies that $0 \neq [\mathrm{Ind}_{C\cap N}^C(\pi(N,A)(\psi)),\, \pi(G,A)(\chi)]_C = [\pi(N,A)(\psi), \mathrm{Res}_{C\cap N}^C(\pi(G,A)(\chi))]_{C\cap N}$. Thus $W(b)$ covers $W(\beta)$ and we are done.

For the final results of this section, we assume that $D \trianglelefteq G$, where D is a p-group. (These results may hold for arbitrary sets of primes.) Clearly if $g \in G$, then D normalizes $gD = Dg$, $G_{p'}$ and $(gD) \cap G_{p'}$.

Lemma 1.22. *Let $g \in G$. Then the following five conditions are equivalent:*

(a) *$gD \in (G/D)_{p'}$,*

(b) *$g_p \in D$,*

(c) *$x_p \in D$ for all $x \in gD$,*

(d) *$(gD) \cap G_{p'} \neq \emptyset$; and*

(e) *$xD = x_{p'}D$ for all $x \in gD$.*

In which case, if $z \in (gD) \cap G_{p'}$, then:

(f) *$(gD) \cap G_{p'} = z^D$;*

(g) *if $\mathcal{T}$ is a right transversal of $C_D(z)$ in D, then $gD = \bigcup_{t\in\mathcal{T}}(zC_D(z))^t$, where the union is disjoint;*

(h) *if $\mathcal{T}$ is a right transversal of $G_D(g_{p'})$ in D, then $gD = \bigcup_{t\in\mathcal{T}}(gC_D(g_{p'}))^t$, where the union is disjoint; and*

(i) *if $\psi : gD \to \mathcal{K}$ is D-stable, then*

$$\sum_{x\in gD}\psi(x) = |D : C_D(g_{p'})|\left(\sum_{d\in C_D(g_{p'})}\psi(gd)\right).$$

Proof. The equivalence of (a)–(e) is well known and easy. Let $z \in (gD)\cap G_{p'}$. Then $z^D \subseteq (gD)\cap G_{p'}$ since D normalizes $(gD)\cap G_{p'}$. Let $u \in (gD)\cap G_{p'} = (zD)\cap G_{p'}$. Then $\langle u\rangle \leq D\langle z\rangle$, $\langle u\rangle$ is a p'-subgroup and $\langle z\rangle$ is a p'-subgroup complement to D in $D\langle z\rangle$. Thus [7, Theorem 6.3.6] yields an element $d \in D$ such that $\langle u\rangle^d \leq \langle z\rangle$.

Hence $u^d \in (zD) \cap \langle z \rangle = \{z\}$ and (f) holds. Let $u \in gD = zD$. Then $u = u_{p'}u_p$, where $u_p \in D$ and $u_{p'} \in (gD) \cap G_{p'} = z^D$. Hence $u_{p'}^d = z$ for some $d \in D$ and $u^d = zu_p^d \in zC_D(z)$. Thus $gD = \bigcup_{d\in D}(zC_D(z))^d = \bigcup_{t\in\mathcal{T}}(zC_D(z))^t$ since $C_D(z)$ normalizes $zC_D(z)$. Since $|\bigcup_{t\in\mathcal{T}}(zC_D(z))^t| \le |D : C_D(z)||zC_D(z)| = |D|$, (g) holds. Note that $g = g_{p'}g_p$, where $g_{p'} \in (gD) \cap G_{p'}$ and $g_p \in C_D(g_{p'})$. Then $gD = \bigcup_{t\in\mathcal{T}}(g_{p'}C_D(g_{p'}))^t = \bigcup_{t\in\mathcal{T}}(gC_D(g_{p'}))^t$ and we are done.

Let $G_{p'} = \bigcup_{i\in I}\mathcal{O}_i$ be the D-conjugation orbit decomposition of $G_{p'}$.

Corollary 1.23. *$\varphi : (G/D)_{p'} \to \{\mathcal{O}_i | i \in I\}$ such that $xD \in (G/D)_{p'} \mapsto (xD)\cap G_{p'}$ is a bijection.*

2. Proofs of Theorems 1 and 2

We begin with a proof of Theorem 1. Under the hypotheses of Theorem 1, let $H = N_G(D)$ and $K = DC_G(D) \trianglelefteq H$. Then Lemma 1.21(b) yields a block $\beta \in B\ell(K)^A$ with defect group D that is covered by $Br_D(b)$. Here β is also a block of $C_G(D)$ and (D, β) is a maximal b-subpair. By [19, Proposition 4(i)], $(D, W(\beta))$ is a maximal $W(b)$-subpair. Here $W(\beta)$ is a block of $DC_C(D) = C_K(A)$ and $W(Br_D(b))$ covers $W(\beta) \in B\ell(C_K(A))$ by Lemma 1.21(c). Since $(D, W(\beta))$ is a maximal $W(b)$-subpair, it follows that $Br_D(W(b))$ is the unique block of $N_C(D)$ that covers $W(\beta) \in B\ell(C_K(A))$. Thus $W(Br_D(b)) = Br_D(W(b))$ and we are done.

Next we proceed to demonstrate Theorem 2. So we assume that $D \trianglelefteq G$. Let $A, G, D \trianglelefteq C = C_G(A)$, $b \in B\ell(G)^A$ and $W(b) \in B\ell(C)$ be as in Theorem 2. Set $L = C_G(D) \trianglelefteq G$ and $K = DC_G(D) \trianglelefteq G$ and note that $D = O_p(G)$. By [10, Theorem 13.27], there is an A-stable block β of K that is covered by b. Since $D = O_p(K)$, [12, Proposition 4.2] implies that D is the defect group of β. Let $T = \mathrm{Stab}_G(\beta)$ so that $K = DC_G(D) \le T \le G$ and T is A-invariant. Set $L_1 = L \cap C = C_C(D) \trianglelefteq C$ and $K_1 = K \cap C = DC_C(D) \trianglelefteq C$, so that $W(\beta) \in B\ell(K_1)$ has defect group D. Also $T \cap C = \mathrm{Stab}_C(\beta) = \mathrm{Stab}_C(W(\beta))$ by Corollary 1.20(c). As is well known, there is a unique block b_T of T that covers β and is such that $\{\mathrm{Ind}_T^G(\psi) \mid \psi \in \mathrm{Irr}(b_T)\} = \mathrm{Irr}(b)$ and b_T induces b via Brauer block induction. Also b_T is A-stable and has defect group D. Moreover, $(\mathcal{O}G)b \cong \mathrm{Ind}_T^G((\mathcal{O}T)b_T)$ as interior G-algebra and as A-algebras, $b = Tr_T^G(b_T)$, and the Brauer categories of $(\mathcal{O}G)b$ and $(\mathcal{O}T)b_T$ are equivalent, and $(\mathcal{O}T)b_T$ and $(\mathcal{O}G)b$ are Morita equivalent $\mathcal{O}$-algebras by a categeorical equivalence that on the character level yields the bijection $\mathrm{Ind}_T^G : \mathrm{Irr}(b_T) \to \mathrm{Irr}(b)$.

Here $W(b) \in B\ell(C)$ covers $W(\beta)$ and both have defect group D, $W(b_T) \in B\ell(T\cap C)$, $W(b_T)$ covers $W(\beta)$ and also has defect group D by [10, Theorem 13.29]. Moreover, $\mathrm{Irr}(W(b)) = \{W(\mathrm{Ind}_T^G(\psi)) \mid \psi \in \mathrm{Irr}(b_T)\} = \{\mathrm{Ind}_{T\cap C}^C(\pi(T, A)(\psi)) \mid \psi \in \mathrm{Irr}(b_T)\}$ by [8, Theorem 2.3(ii)]. Thus $(\mathcal{O}C)W(b) \cong \mathrm{Ind}_{C\cap T}^C((\mathcal{O}(C \cap T))W(b_T))$ as interior C-algebras, $W(b) = Tr_{C\cap T}^C(W(b_T))$, and the Brauer categories of $(\mathcal{O}C)W(b)$ and $(\mathcal{O}(C \cap T))W(b_T)$ are equivalent, and $W(b_T)$ induces $W(b)$ via Brauer block induction, and $\mathcal{O}(C \cap T)W(b_T)$ and $(\mathcal{O}C)W(b)$ are Morita equivalent $\mathcal{O}$-algebras via a categorical equivalence that on the character level yields the bijection $\mathrm{Ind}_{C\cap T}^C : \mathrm{Irr}(W(b_T)) \to \mathrm{Irr}(W(b))$.

Consequently it suffices to assume that β is G-stable. In that case, $\beta = b$ and b is a block idempotent of $\mathcal{O}G$, $\mathcal{O}K$ and $\mathcal{O}L$, where $Z(D)$ is a defect group of b as a block of $\mathcal{O}L$. Also $|G/K| = m$ is relatively prime to p. Similarly $W(\beta) = W(b)$

is a block idempotent of $\mathcal{O}C, \mathcal{O}K_1$ and $\mathcal{O}L_1$ where $Z(D)$ is a defect group of $W(b)$ as a block of $\mathcal{O}L_1$ (cf. [5, V, Sections 3 and 4]).

Since $[G, A] \leq C_G(D)$, we have $G = C_G(D)C$. Since $DC_C(D)/C_C(D)$ is a normal Sylow p-subgroup of $C/C_C(D)$, there is a subgroup H_1 of C with $C_C(D) \leq H_1$, $C = H_1D$ and $H_1 \cap D = Z(D)$. Thus the map $hC_C(D) \mapsto hK$ for $h \in H_1$ is an isomorphism of H_1/L_1 onto G/K. Let $\mathcal{T}$ be a transversal of L_1 in H_1 with $1 \in \mathcal{T}$, so that $\mathcal{T}$ is a transversal of K in G and $|\mathcal{T}| = |H_1/L_1| = |G/K| = |C/K_1| = m$.

Set $H = C_G(D)H_1$, so that $L \leq H \leq G = HD$ and $H \cap D = Z(D)$, H is A-invariant, $C_H(A) = C \cap H = H_1$ and $H_1 \cap C_G(D) = C_C(D)$. As $L_1 = C_{H_1}(D)$, H_1/L_1 acts faithfully by conjugation on D. Let $\mathcal{N} = D \rtimes (H_1/L_1)$.

As is well known, the block b of $\mathcal{O}K$ contains exactly one irreducible character θ such that $D \leq \mathrm{Ker}(\theta)$ and b contains exactly one irreducible Brauer character φ and $\varphi(x) = \theta(x)$ for all p'-elements x of K. Let V be an $\mathcal{O}$-free $\mathcal{O}K$-module that affords θ and let $r = \theta(1) = \mathrm{rank}(V/\mathcal{O})$. Then $D \leq \mathrm{Ker}(V)$ and $\bar{V}$ is an irreducible kK-module in b with Brauer character φ. Let $P(\bar{V})$ denote a projective indecomposable $(\mathcal{O}K)b$-module corresponding to $\bar{V}$ and let Φ be the character of $P(\bar{V})$. Then $\Phi(x) = |D|\varphi(x) = |D|\theta(x)$ for all p'-elements x of K by [5, V, Corollary 4.6] and hence $\mathrm{rank}(P(\bar{V})/\mathcal{O}) = r|D|$. Since $(\mathcal{O}K)b \cong P(\bar{V})^r$ in $(\mathcal{O}K)b$-mod, we conclude that $\mathrm{rank}((\mathcal{O}K)b/\mathcal{O}) = r^2|D|$. Note that $\mathrm{Res}^K_L(\theta)$ is the unique irreducible character in the block b of L with $Z(D) \leq \mathrm{Ker}(\mathrm{Res}^K_L(\theta))$; we similarly conclude that $\mathrm{rank}((\mathcal{O}L)b/\mathcal{O}) = r^2|Z(D)|$. Also $\bar{\theta} = \mathrm{char}(\bar{V})$ is the unique irreducible character of $(kK)\bar{b}$, and $\mathrm{Res}^K_L(\bar{\theta}) = \mathrm{char}(\mathrm{Res}^K_L(\bar{V}))$ is the unique irreducible character of $(kL)\bar{b}$ and $r_p = |K/D|_p = |G/D|_p$.

Clearly $(\mathcal{O}L)b$, $(\mathcal{O}K)b$ and $(\mathcal{O}H)b$ are $\mathcal{O}$-subalgebras of $(\mathcal{O}G)b$.

Let $\theta_1, \varphi_1, V_1, r_1 = \theta_1(1)$, $\bar{V}_1, P(\bar{V}_1)$, Φ_1 be the corresponding objects of the block $W(b)$ of K_1 with defect group D. Note that $\pi(K, A)(\theta) = \theta_1$ since $D \leq \ker(\pi(K, A)(\theta))$ and $\pi(K, A)(\theta) \in \mathrm{Irr}_k(W(b))$.

Proposition 2.1. (a) *the $\mathcal{O}$-algebra homomorphism $M(b) : \mathcal{O}D \to (\mathcal{O}K)b$ such that $\alpha \mapsto \alpha b$ for all $\alpha \in \mathcal{O}D$ is a G-injection such that $M(b)(\mathcal{O}Z(D)) = (\mathcal{O}Z(D))b \leq (\mathcal{O}L)b$;*

(b) *there is an $\mathcal{O}$-simple subalgebra $\mathcal{S}$ of $(\mathcal{O}L)b$ such that $b \in \mathcal{S}$, $\mathcal{S} \cong M_r(\mathcal{O})$ as $\mathcal{O}$-algebras, $(\mathcal{O}L)b = \mathcal{S} + J((\mathcal{O}L)b)$, $(\mathcal{O}K)b = \mathcal{S} + J((\mathcal{O}K)b)$, $\mathcal{S}$ is a maximal $\mathcal{O}$-semi-simple $\mathcal{O}$-subalgebra of both $(\mathcal{O}L)b$ and $(\mathcal{O}K)b$, and $J(\mathcal{O})\mathcal{S} = \mathcal{S} \cap J((\mathcal{O}K)b) = \mathcal{S} \cap J((\mathcal{O}L)b)$;*

(c) *$C_{(\mathcal{O}K)b}(\mathcal{S}) = (\mathcal{O}D)b$ and $C_{(\mathcal{O}L)b}(\mathcal{S}) = (\mathcal{O}Z(D))b$ and the $\mathcal{O}$-linear "multiplication maps" $\mu : \mathcal{S} \otimes_{\mathcal{O}} ((\mathcal{O}D)b) \to (\mathcal{O}K)b$ such that $s \otimes_{\mathcal{O}} \alpha \mapsto s\alpha$ for all $s \in \mathcal{S}$ and all $\alpha \in (\mathcal{O}D)b$, and $\mu : \mathcal{S} \otimes (\mathcal{O}Z(D))b \to (\mathcal{O}L)b$ such that $s \otimes_{\mathcal{O}} \alpha \mapsto s\alpha$ for all $s \in \mathcal{S}$ and all $\alpha \in (\mathcal{O}Z(D))b$ are $\mathcal{O}$-algebra isomorphisms;*

(d) *$(\mathcal{O}G)b = \oplus_{t \in \mathcal{T}}((\mathcal{O}K)b)(tb)$ and $(\mathcal{O}H)b = \oplus_{t \in \mathcal{T}}((\mathcal{O}L)b)(tb)$ exhibit $(\mathcal{O}G)b$ and $(\mathcal{O}H)b$ as H_1/L_1-crossed product $\mathcal{O}$-algebras with $((\mathcal{O}G)b)_{tL_1} = ((\mathcal{O}K)b)(tb)$ and $((\mathcal{O}H)b)_{tL_1} = ((\mathcal{O}L)b)(tb)$ for all $t \in \mathcal{T}$, respectively;*

(e) *for each $t \in \mathcal{T}$, there is an element $w_t \in ((\mathcal{O}L)b)^\times$ such that*

$$v_t = w_t(tb) \in C_{((\mathcal{O}L)b)(tb)}(\mathcal{S}) \cap ((\mathcal{O}H)b)^\times, v_1 = w_1 = b,$$
$$C_{(OH)b}(\mathcal{S}) = \oplus_{t \in \mathcal{T}}(((\mathcal{O}Z(D))b)v_t),$$
$$C_{(\mathcal{O}G)b}(\mathcal{S}) = \oplus_{t \in \mathcal{T}}(((\mathcal{O}D)b)v_t)$$

and $v_t \alpha v_t^{-1} = t\alpha t^{-1} = {}^t\alpha$ for all $\alpha \in (\mathcal{O}D)b$, $v_t \alpha v_t^{-1} = w_t({}^t\alpha)w_t^{-1}$ for all $\alpha \in (\mathcal{O}K)b$ and $v_t \in N_{((\mathcal{O}G)b)^\times}((\mathcal{O}K)b) \cap N_{((\mathcal{O}G)b)^\times}((\mathcal{O}L)b)$ for all $t \in \mathcal{T}$. Also $\mathcal{A} =$

$\oplus_{t\in\mathcal{T}}\mathcal{O}v_t$ is a twisted H_1/L_1-group $\mathcal{O}$-subalgebra of $(\mathcal{O}H)b$ with associated $c \in Z^2(H_1/L_1,\ f(k^\times))$ such that $c^m = 1$ and $c(tL_1, L_1) = c(L_1, tL_1) = 1$ for all $t \in \mathcal{T}$, $\Omega = \{c(t_1L_1, t_2L_1) \mid t_1, t_2 \in \mathcal{T}\}$ is a finite subgroup of $\mathcal{O}^\times$ of order n dividing m and $\{\omega v_t \mid \omega \in \Omega,\ t \in \mathcal{T}\}$ is a subgroup of $\mathcal{A}^\times$ of order mn;

(f) *multiplication $\mu : \mathcal{S} \otimes_{\mathcal{O}} C_{(\mathcal{O}G)b}(\mathcal{S}) \to (\mathcal{O}G)b$ such that $s \otimes \alpha \mapsto s\alpha$ for all $s \in \mathcal{S}$ and all $\alpha \in C_{(\mathcal{O}G)b}(\mathcal{S})$ is an $\mathcal{O}$-algebra isomorphism;*

(g) *$(\mathcal{O}H)bJ((\mathcal{O}L)b)$ is an H_1/L_1-graded ideal of $(\mathcal{O}H)b$ and hence $\widetilde{(\mathcal{O}H)b} = (\mathcal{O}H)b/((\mathcal{O}H)bJ((\mathcal{O}L)b))$ is an H_1/L_1-crossed product k-algebra with $(\widetilde{(\mathcal{O}H)b})_{tL_1} = [((\mathcal{O}L)b)(tb) + (\mathcal{O}H)bJ((\mathcal{O}L)b)]/[(\mathcal{O}H)bJ((\mathcal{O}L)b)]$ for all $t \in \mathcal{T}$ and $(\widetilde{(\mathcal{O}H)b})_{L_1} \cong \mathcal{S}/J(\mathcal{O})\mathcal{S} \cong M_r(k)$ as k-algebras. Moreover, the injection $i : \mathcal{A} = \oplus_{t\in\mathcal{T}}\mathcal{O}v_t \to (\mathcal{O}H)b$ induces an H_1/L_1-graded isomorphism of $\bar{\mathcal{A}} = \mathcal{A}/J(\mathcal{O})\mathcal{A}$ onto the associated "Clifford extension" of $(\mathcal{O}H)b$ with respect to $(\mathcal{O}H)bJ((\mathcal{O}L)b)$; and*

(h) *let $\mathcal{L}$ denote the $\mathcal{O}$-free twisted $\mathcal{N}$-group algebra with $\mathcal{O}$-free basis $\{(d, tL_1) \mid d \in D,\ t \in \mathcal{T}\}$ such that $(d_1, t_1L_1)(d_2, t_2L_1) = c(t_1L_1, t_2L_1)(d_1({}^{t_1}d_2), t_3L_1)$, where $d_1, d_2 \in D$ and $t_1, t_2 \in \mathcal{T}$ and $(t_1L_1)(t_2L_1) = t_3L_1$ for a unique $t_3 \in \mathcal{T}$. Then $\mathcal{L}$ is $\mathcal{O}$-algebra isomorphic to $C_{(\mathcal{O}G)b}(\mathcal{S}) = \oplus_{t\in\mathcal{T}}(((\mathcal{O}D)b)v_t)$ via the $\mathcal{O}$-linear map such that $(d, tL_1) \mapsto (db)v_t$ for all $d \in D$ and all $t \in \mathcal{T}$.*

Remark 2.2. The "Clifford extension" of Proposition 2.1(g) is the "Clifford extension" of kH with respect to the irreducible character $\mathrm{Res}^K_L(\bar{\theta}) = \mathrm{char}(\mathrm{Res}^K_L(\bar{V}))$ of kL.

Proof. Clearly Lemma 1.17 yields (a). Since φ is the unique irreducible Brauer character of $b \in B\ell(\mathcal{O}K)$ and $\mathrm{Res}^K_L(\varphi)$ is the unique irreducible Brauer character of $b \in B\ell(\mathcal{O}L)$, Propositions 1.12 and 1.15 and [18, Theorem 7.3] yield (b). Here the $\mathcal{O}$-linear "multiplication" map $\mu : \mathcal{S} \otimes C_{(\mathcal{O}K)b}(\mathcal{S}) \to (\mathcal{O}K)b$ is an $\mathcal{O}$-algebra isomorphism by [18, Proposition 7.5]. Thus $\mathrm{rank}(C_{(\mathcal{O}K)b}(\mathcal{S})/\mathcal{O}) = |D| = \mathrm{rank}((\mathcal{O}D)b/\mathcal{O})$ and $(\mathcal{O}D)b \leq C_{(\mathcal{O}K)b}(\mathcal{S})$. Also "reducing mod p", we similarly have $\dim((kD)\bar{b}/k) = |D| = \dim(C_{kK\bar{b}}(\bar{\mathcal{S}})/k)$. Thus $C_{kK\bar{b}}(\bar{\mathcal{S}}) = (kD)\bar{b}$ and $(\mathcal{O}D)b \leq C_{(\mathcal{O}K)b}(\mathcal{S}) \leq (\mathcal{O}D)b + J(\mathcal{O})(\mathcal{O}K)b$. Let $0 \neq x \in J(\mathcal{O})(\mathcal{O}K)b \cap C_{(\mathcal{O}K)b}(\mathcal{S})$. Then there are elements $0 \neq j \in J(\mathcal{O})$ and $0 \neq u \in (\mathcal{O}K)b$ such that $x = ju$ and we readily conclude that $u \in C_{(\mathcal{O}K)b}(\mathcal{S})$. Thus $(\mathcal{O}D)b \leq C_{(\mathcal{O}K)b}(\mathcal{S}) \leq (\mathcal{O}D)b + J(\mathcal{O})C_{(\mathcal{O}K)b}(\mathcal{S})$ and Nakayama's Lemma implies that $C_{(\mathcal{O}K)b}(\mathcal{S}) = (\mathcal{O}D)b$. Similarly $C_{(\mathcal{O}L)b}(\mathcal{S}) = (\mathcal{O}Z(D))b$ and (c) holds. Clearly (d) holds.

For (e), recall that $1 \in \mathcal{T}$ and that $((\mathcal{O}L)b)^\times = \mathcal{S}^\times(b + J((\mathcal{O}L)b)) = (b + J(\mathcal{O}L)b))\mathcal{S}^\times$ by Proposition 1.12.

For each $t \in \mathcal{T}$, there is an element $w'_t \in ((\mathcal{O}L)b)^\times$ such that

$$w'_t(tb) \in C_{(\mathcal{O}L)b(tb)}(\mathcal{S}) \cap ((\mathcal{O}H)b)^\times,$$

where we may assume that $w'_1 = b$ if $t = 1$ by Proposition 1.15(b). Consequently $(\mathcal{O}H)b = \oplus_{t\in\mathcal{T}}(\mathcal{O}L)b((w'_t(tb))$ and $C_{(\mathcal{O}H)b}(\mathcal{S}) = \oplus_{t\in\mathcal{T}}\mathcal{O}Z(D)b(w'_t(tb))$, where $\mathcal{O}Z(D)b$ is a commutative $\mathcal{O}$-algebra. Lemma 1.13 implies that there is a set $\{u_t | t \in \mathcal{T}\} \subseteq (\mathcal{O}Z(D)b)^\times$ such that $\oplus_{t\in\mathcal{T}}\mathcal{O}(u_tw'_t(tb))$ is a twisted H_1/L_1-group $\mathcal{O}$-algebra with associated $c \in Z^2(H_1/L_1, f(k^\times))$. Then, since $k^\times$ is $m = |H_1/L_1|$-divisible, Lemma 1.4 implies that we may replace each u_t by an element of $\mathcal{O}^\times u_t$ to assure that $c^m = 1$. Then, referring to Lemma 1.2, we may also assume that $c(tL_1, L_1) = c(L_1, tL_1) = 1$ for all $t \in \mathcal{T}$. Then $(u_1b)(u_1b) = (u_1b)$ and hence $u_1 = b$ and with $w_t = u_tw'_t$ for all $t \in \mathcal{T}$ we have (e).

Finally [18, Proposition 7.5] implies (f), and (g) and (h) are clear.

For similar reasons, we have:

Proposition 2.3. (a) *the $\mathcal{O}$-algebra homomorphism $M(W(b))\colon \mathcal{O}D \to (\mathcal{O}K_1)W(b)$ such that $\alpha \mapsto \alpha W(b)$ for all $\alpha \in \mathcal{O}D$ is a C-injection such that $M(W(b))(\mathcal{O}Z(D)) = (\mathcal{O}Z(D))W(b) \le (\mathcal{O}L_1)W(b)$,*

(b) *there is an $\mathcal{O}$-simple subalgebra $\mathcal{S}_1$ of $(\mathcal{O}L_1)W(b)$ such that $W(b) \in \mathcal{S}_1$, $\mathcal{S}_1 \cong M_{r_1}(\mathcal{O})$ as $\mathcal{O}$-algebras, $(\mathcal{O}L_1)W(b) = \mathcal{S}_1 + J((\mathcal{O}L_1)W(b))$, $(\mathcal{O}K_1)W(b) = \mathcal{S}_1 + J((\mathcal{O}K_1)W(b))$, $\mathcal{S}_1$ is a maximal $\mathcal{O}$-semi-simple $\mathcal{O}$-subalgebra of both $(\mathcal{O}L_1)W(b)$ and $(\mathcal{O}K_1)W(b)$ and $J(\mathcal{O})\mathcal{S}_1 = \mathcal{S}_1 \cap J((\mathcal{O}K_1)W(b)) = \mathcal{S}_1 \cap J((\mathcal{O}L_1)W(b))$;*

(c) *$C_{(\mathcal{O}K_1)W(b)}(\mathcal{S}_1) = (\mathcal{O}D)W(b)$ and $C_{(\mathcal{O}L_1)W(b)}(\mathcal{S}_1) = (\mathcal{O}Z(D))W(b)$ and the $\mathcal{O}$-linear "multiplication maps" $\mu : \mathcal{S}_1 \otimes_{\mathcal{O}} ((\mathcal{O}D)W(b)) \to (\mathcal{O}K_1)W(b)$ such that $s_1 \otimes_{\mathcal{O}} \alpha \mapsto s_1\alpha$ for all $s_1 \in \mathcal{S}_1$ and all $\alpha \in (\mathcal{O}D)W(b)$, and $\mu : \mathcal{S}_1 \otimes_{\mathcal{O}} ((\mathcal{O}Z(D))W(b)) \to (\mathcal{O}L_1)W(b)$ such that $s_1 \otimes_{\mathcal{O}} \alpha \mapsto s_1\alpha$ for all $s_1 \in \mathcal{S}_1$ and all $\alpha \in (\mathcal{O}Z(D))W(b)$ are $\mathcal{O}$-algebra isomorphisms;*

(d)
$$(\mathcal{O}C)W(b) = \oplus_{t\in\mathcal{T}}((\mathcal{O}K_1)W(b))(tW(b))$$
and
$$(\mathcal{O}H_1)W(b) = \oplus_{t\in\mathcal{T}}((\mathcal{O}L_1)W(b))(tW(b))$$
exhibit $(\mathcal{O}C)W(b)$ and $(\mathcal{O}H_1)W(b)$ as H_1/L_1-crossed product $\mathcal{O}$-algebras with
$$((\mathcal{O}C)W(b))_{tL_1} = ((\mathcal{O}K_1)W(b))(tW(b))$$
and
$$((\mathcal{O}H_1)W(b))_{tL_1} = ((\mathcal{O}L_1)W(b))(tW(b))$$
for all $t \in \mathcal{T}$;

(e) *for each $t \in \mathcal{T}$, there is an element $w'_t \in ((\mathcal{O}L_1)W(b))^\times$ such that*
$$v'_t = w'_t(tW(b)) \in C_{(\mathcal{O}L_1)W(b)(tW(b))}(\mathcal{S}_1) \cap ((\mathcal{O}H_1)W(b))^\times, \quad v'_1 = w'_1 = W(b),$$
$$C_{(\mathcal{O}H_1)W(b)}(\mathcal{S}_1) = \oplus_{t\in\mathcal{T}}(((\mathcal{O}Z(D))W(b))v'_t),$$
$$C_{(\mathcal{O}C)W(b)}(\mathcal{S}_1) = \oplus_{t\in\mathcal{T}}(((\mathcal{O}D)W(b))v'_t),$$
$v'_t\alpha(v'_t)^{-1} = t\alpha t^{-1} = {}^t\alpha$ for all $\alpha \in (\mathcal{O}D)W(b)$, $v'_t\alpha(v'_t)^{-1} = (w'_t)({}^t\alpha)(w'_t)^{-1}$ for all $\alpha \in (\mathcal{O}K_1)W(b)$ and $v'_t \in N_{((\mathcal{O}C)W(b))^\times}((\mathcal{O}K_1)W(b)) \cap N_{((\mathcal{O}C)W(b))^\times}((\mathcal{O}L_1)W(b))$ for all $t \in \mathcal{T}$. Also $\mathcal{A}' = \oplus_{t\in\mathcal{T}}\mathcal{O}v'_t$ is a twisted H_1/L_1-group $\mathcal{O}$-subalgebra of $(\mathcal{O}H_1)W(b)$ with associated $c' \in Z^2(H_1/L_1, f(k^\times))$ such that $(c')^m = 1$ and $c'(tL_1, L_1) = c'(L_1, tL_1) = 1$ for all $t \in \mathcal{T}$;

(f) *multiplication*
$$\mu_1 : \mathcal{S}_1 \otimes C_{(\mathcal{O}C)W(b)}(\mathcal{S}_1) \to (\mathcal{O}C)W(b)$$
such that $s_1 \otimes \alpha \mapsto s_1\alpha$ for all $s_1 \in \mathcal{S}_1$ and all $\alpha \in C_{(\mathcal{O}C)W(b)}(\mathcal{S}_1)$ is an $\mathcal{O}$-algebra isomorphism;

(g) *$(\mathcal{O}H_1)W(b)J((\mathcal{O}L_1)W(b))$ is an H_1/L_1-graded ideal of $(\mathcal{O}H_1)W(b)$ and hence $(\widetilde{\mathcal{O}H_1})W(b) = ((\mathcal{O}H_1)W(b))/((\mathcal{O}H_1)W(b)J((\mathcal{O}H_1)W(b)))$ is an H_1/L_1-crossed product k-algebra with*
$$((\widetilde{\mathcal{O}H_1})W(b))_{tL_1} = [((\mathcal{O}L_1)W(b)(tb) + (\mathcal{O}H_1)W(b)J((\mathcal{O}L_1)W(b))] / [(\mathcal{O}H_1)W(b)J((\mathcal{O}L_1)W(b))]$$

for all $t \in \mathcal{T}$ *and* $\widetilde{(\mathcal{O}H_1)W(b)}_{L_1} \cong \mathcal{S}_1/(J(\mathcal{O})\mathcal{S}_1) \cong M_{r_1}(k)$ *as* k*-algebras. Moreover, the injection* $i' : \mathcal{A}' = \oplus_{t\in\mathcal{T}}\mathcal{O}v'_t \to (\mathcal{O}H_1)W(b)$ *induces an* H_1/L_1*-graded isomorphism of* $\mathcal{A}' = \mathcal{A}'/(J(\mathcal{O})\mathcal{A}')$ *onto the associated "Clifford extension" of* $(\mathcal{O}H_1)W(b)$ *with respect to* $(\mathcal{O}H_1)W(b)J((\mathcal{O}L_1)W(b))$*; and*

(h) *let* $\mathcal{L}'$ *denote the* $\mathcal{O}$*-free twisted* $\mathcal{N}$*-group algebra with* $\mathcal{O}$*-free basis* $\{(d, tL_1) \mid d \in D,\ t \in \mathcal{T}\}$ *such that* $(d_1, t_1L_1)(d_2, t_2L_1) = c'(t_1L_1, t_2L_1)(d_1({}^{t_1}d_2), t_3L_1)$*, where* $d_1, d_2 \in D$ *and* $t_1, t_2 \in \mathcal{T}$ *and* $(t_1L_1)(t_2L_1) = t_3L_1$ *for a unique* $t_3 \in \mathcal{T}$*. Then* $\mathcal{L}'$ *is* $\mathcal{O}$*-algebra isomorphic to* $C_{(\mathcal{O}C)W(b)}(\mathcal{S}_1) = \oplus_{t\in\mathcal{T}}((\mathcal{O}D)W(b))v'_t$ *via the* $\mathcal{O}$*-linear map such that* $(d, tL_1) \mapsto dW(b)v'_t$ *for all* $d \in D$ *and all* $t \in \mathcal{T}$*.*

Remark 2.4. The "Clifford extension" of Proposition 2.3(g) is the "Clifford extension" of kH_1 with respect to the irreducible character $\mathrm{Res}^{K_1}_{L_1}(\bar{\theta}_1) = \mathrm{char}(\mathrm{Res}^{K_1}_{L_1}(\bar{V}_1))$ of kL_1.

By Propositions 2.1 and 2.3, a Morita equivalence will follow from the proof that the $\mathcal{O}$-free twisted H_1/L_1-group algebras $\mathcal{A}$ and $\mathcal{A}'$ are H_1/L_1-graded isomorphic $\mathcal{O}$-algebras. Applying Propositions 2.1 and 2.3, Remarks 2.2 and 2.4 and Lemma 1.13(b), it suffices to prove that the associated "Clifford extensions" of $(\mathcal{O}H)b$ with respect to $(\mathcal{O}H)bJ((\mathcal{O}L)b)$ and of $(\mathcal{O}H_1)W(b)$ with respect to $(\mathcal{O}H_1)W(b)J((\mathcal{O}L_1)W(b))$ are isomorphic.

At this point, we observe that it suffices to assume that A is cyclic of prime order q.

Consequently, we can apply the proof of [14, Proposition 3.3] that is a consequence of [14, Lemma 3.2] of E.C. Dade working over k as follows.

Let $\tilde{H} = H \rtimes A$ and $\tilde{L} = L \rtimes A$, so that $L \trianglelefteq \tilde{H}$ and $\tilde{L} \trianglelefteq \tilde{H}$ since $H = LH_1$ and A centralizes H_1. We note here that $(\mathcal{K}, \mathcal{O}, k)$ is "big enough" for all subgroups of $\tilde{H}$. As $\gamma = \mathrm{Res}^K_L(\theta) \in \mathrm{Irr}(L)^{\tilde{H}}$ and $(|L|, |A|) = 1$, γ has a unique canonical extension $\tilde{\gamma} \in \mathrm{Irr}(\tilde{L})^{\tilde{H}}$ such that $A \le \mathrm{Ker}(\det(\tilde{\gamma}))$. Then $\bar{\gamma} = \mathrm{Res}^K_L(\bar{\theta})$ is the unique irreducible character of $(kL)\bar{b}$ and $\bar{\tilde{\gamma}}$ is an $\tilde{H}$-stable irreducible character of $k\tilde{L}$ that extends $\bar{\gamma}$. As in [14, Lemma 3.2], since $\tilde{H} = \tilde{L}H$, [3, Theorem 4.4] implies that the "Clifford extensions" of kH with respect to $\bar{\gamma} = \mathrm{Res}^K_L(\bar{\theta})$ and of $k\tilde{H}$ with respect to $\bar{\tilde{\gamma}}$ are isomorphic twisted H_1/L_1-graded k-algebras.

Note that $C_H(A) = H_1$ and let $\tilde{H}_1 = H_1 \times A$ and $\tilde{L}_1 = L_1 \times A$, so that $L_1 \trianglelefteq \tilde{H}_1$ and $\tilde{L}_1 \trianglelefteq \tilde{H}_1$. Set $\delta = \mathrm{Res}^{K_1}_{L_1}(\theta_1) \in \mathrm{Irr}(L_1)^{\tilde{H}_1}$. Then δ has q different extensions to irreducible characters $\delta \times \lambda \in \mathrm{Irr}(\tilde{L}_1)$ for all $\lambda \in \mathrm{Irr}(A)$. Similarly $\bar{\delta}$ is the unique irreducible character of $kL_1\overline{W(b)}$ and $\overline{\delta \times \lambda}$ is an $\tilde{H}_1$-stable irreducible character of $k\tilde{L}_1$ that extends $\bar{\delta}$ for each $\lambda \in \mathrm{Irr}(A)$.

As above, the "Clifford extensions" of kH_1 with respect to $\bar{\delta}$ and of $k\tilde{H}_1$ with respect to $\overline{\delta \times \lambda}$ are isomorphic twisted H_1/L_1-graded k-algebras for each $\lambda \in \mathrm{Irr}(A)$.

Clearly $\gamma = \mathrm{Res}^K_L(\theta) \in \mathrm{Irr}(b)$, where $b \in B\ell(L)$ has defect group $Z(D)$ and $Z(D) \le \mathrm{Ker}(\gamma)$. Also $\tilde{\gamma}$ lies in a block of $\tilde{L}$ with defect group $Z(D)$ that covers b and $Z(D) \le \mathrm{Ker}(\tilde{\gamma})$. Similarly $\delta = \mathrm{Res}^{K_1}_{L_1}(\theta_1) \in \mathrm{Irr}(W(b))$, where $W(b) \in B\ell(L_1)$ has defect group $Z(D)$, $Z(D) \le \mathrm{Ker}(\delta)$ and $\delta \times \lambda$ lies in a block of $\tilde{L}_1$ with defect group $Z(D)$ that covers $W(b)$ and $Z(D) \le \mathrm{Ker}(\delta \times \lambda)$ for all $\lambda \in \mathrm{Irr}(A)$.

Fix $\lambda \in \mathrm{Irr}(A)$.

We may view $\tilde{\gamma}$ and $\delta \times \lambda$ as elements of $\mathrm{Irr}(\tilde{L}/Z(D))$ and $\mathrm{Irr}(\tilde{L}_1/Z(D))$, resp.; in which case we have

$$(2.1) \qquad (\mathrm{Res}^{\tilde{L}}_{\tilde{L}_1}(\tilde{\gamma}),\ \delta \times \lambda)_{\tilde{L}_1} = (\mathrm{Res}^{\tilde{L}/Z(D)}_{\tilde{L}_1/Z(D)}(\tilde{\gamma}), \delta \times \lambda)_{\tilde{L}_1/Z(D)}.$$

Let $\tilde{V}$ be an $\mathcal{O}$-free indecomposable $\mathcal{O}\tilde{L}$-module that affords $\tilde{\gamma}$ and let $\tilde{W}$ be an $\mathcal{O}$-free indecomposable $\mathcal{O}\tilde{L}_1$-module that affords $\delta \times \lambda$. Here $Z(D) \leq \mathrm{Ker}(\tilde{V})$ and $\tilde{V}$ lies in a block of $\tilde{L}$ with defect group $Z(D)$ so that $\bar{\tilde{V}}$ is an irreducible $k\tilde{L}$-module with character $\bar{\tilde{\gamma}}$. Moreover, we may view $\bar{\tilde{V}}$ as an irreducible $k(\tilde{L}/Z(D))$-module that lies in a block of defect 0 of $\tilde{L}/Z(D)$. Similarly, $Z(D) \leq \mathrm{Ker}(\tilde{W})$ and $\tilde{W}$ lies in a block of $\tilde{L}_1$ with defect group $Z(D)$ so that $\bar{\tilde{W}}$ is an irreducible $k\tilde{L}_1$-module with character $\overline{\delta \times \lambda}$. Clearly we may view $\bar{W}$ as an irreducible $k(L_1/Z(D))$-module that lies in a block of defect 0 of $L_1/Z(D)$. Here the multiplcity ω of $\bar{\tilde{W}}$ as a $k\tilde{L}_1$-module composition factor of $\mathrm{Res}^{\tilde{L}}_{\tilde{L}_1}(\bar{\tilde{V}})$ is the same as the multiplicity of $\bar{\tilde{W}}$ as a $k(\tilde{L}_1/Z(D))$-module composition factor of $\mathrm{Res}^{\tilde{L}/Z(D)}_{\tilde{L}_1/Z(D)}(\bar{\tilde{V}})$. Now Lemma 1.18 and (2.1) imply that $\omega = (\mathrm{Res}^{\tilde{L}}_{\tilde{L}_1}(\tilde{\gamma}), \delta \times \lambda)_{\tilde{L}_1}$. Consequently, the proof of [14, Lemma 3.2] implies that we may choose $w'_t \in ((\mathcal{O}L_1)W(b))^\times$ for each $t \in \mathcal{T}$, where $w'_1 = W(b)$ such that, setting $v'_t = tW(b)w'_t$ for all $t \in \mathcal{T}$, we have

$$(2.2) \qquad C_{(\mathcal{O}C)W(b)}(\mathcal{S}_1) = \oplus_{t\in\mathcal{T}}(((\mathcal{O}D)W(b))v'_t),$$

$v'_t \alpha v'_t = t\alpha t^{-1} = {}^t\alpha$ for all $\alpha \in (\mathcal{O}D)W(b)$, and such that there is an $\mathcal{O}$-algebra isomorphism $\Phi : C_{(\mathcal{O}C)W(b)}(\mathcal{S}_1) = \oplus_{t\in\mathcal{T}}(((\mathcal{O}D)W(b))v'_t) \to C_{(\mathcal{O}G)b}(\mathcal{S}) = \oplus_{t\in\mathcal{T}}(((\mathcal{O}D)b)v_t)$ such that $(\alpha W(b))v'_t \mapsto (\alpha b)v_t$ for all $\alpha \in \mathcal{O}D$ and all $t \in \mathcal{T}$. Consequently with this choice, we have $c'(t_1L_1, t_2L_1) = c(t_1L_1, t_2L_1)$ for all $t_1, t_2 \in \mathcal{T}$. Now Propositions 2.1(e), (f) and 2.3(e), (f) imply that $\mathcal{O}Gb$-mod and $\mathcal{O}CW(b)$-mod are Morita equivalent.

Let $\mathcal{S} = \mathrm{End}_\mathcal{O}(U)$ and $\mathcal{S}_1 = \mathrm{End}_\mathcal{O}(U_1)$, where U, U_1 are $\mathcal{O}$-free modules of ranks r, r_1, respectively. Then, as in [8, Section 4], since $\mu : \mathcal{S} \otimes_\mathcal{O} C_{(\mathcal{O}G)b}(\mathcal{S}) \to (\mathcal{O}G)b$ and $\mu_1 : \mathcal{S}_1 \otimes C_{(\mathcal{O}C)W(b)}(\mathcal{S}_1) \to (\mathcal{O}C)W(b)$ of Propositions 2.1(f) and 2.3(f) are $\mathcal{O}$-algebra isomorphisms, the $(\mathcal{O}G)b$-mod-$(\mathcal{O}C)W(b)$ bimodule M inducing the Morita equivalence above is given explicitly by

$$M = (U \otimes_\mathcal{O} (C_{(\mathcal{O}G)b}(\mathcal{S})_\Phi)) \otimes_{C_{(\mathcal{O}C)W(b)}(\mathcal{S}_1)} (C_{(\mathcal{O}C)W(b)}(\mathcal{S}_1) \otimes_\mathcal{O} U_1^*),$$

where $\Phi : C_{(\mathcal{O}C)W(b)}(\mathcal{S}_1) \to C_{(\mathcal{O}G)b}(\mathcal{S})$ is the $\mathcal{O}$-algebra isomorphism above and $U_1^* = \mathrm{Hom}_\mathcal{O}(U_1, \mathcal{O})$ is the dual module of U_1.

Thus $M \cong U \otimes_\mathcal{O} (C_{(\mathcal{O}G)b}(\mathcal{S})_\Phi) \otimes_\mathcal{O} U_1^*$ in $(\mathcal{O}G)b$-mod-$(\mathcal{O}C)W(b)$.

Using the method of [8, Section 4], we show that the indecomposible $(\mathcal{O}G) \otimes_\mathcal{O} (\mathcal{O}C)$-module $\mathcal{M} = U \otimes_\mathcal{O} (C_{(\mathcal{O}G)b}(\mathcal{S})_\Phi) \otimes_\mathcal{O} U_1^*$ has ΔD as a vertex and a trivial source.

Here $D \times D$ is the defect group of the block corresponding to $b \otimes_\mathcal{O} W(b)$ in $(\mathcal{O}G) \otimes_\mathcal{O} (\mathcal{O}C) \cong \mathcal{O}(G \times C)$. Thus $\mathcal{M}$, viewed in $(\mathcal{O}G) \otimes_\mathcal{O} (\mathcal{O}C)$-mod, is $D \times D$-projective. Following [8, Section 4] and noting that $D \trianglelefteq G$, that the isomorphism $\mu : \mathcal{S} \otimes C_{(\mathcal{O}G)b}(\mathcal{S}) \to (\mathcal{O}G)b$ sends $b \otimes_\mathcal{O} db \to db$ for all $d \in D$ and that the isomorphism $\mu_1 : \mathcal{S}_1 \otimes_\mathcal{O} C_{(\mathcal{O}C)W(b)}(\mathcal{S}_1) \to (\mathcal{O}C)W(b)$ sends $W(b) \otimes dW(b) \mapsto dW(b)$ for all $d \in D$, we observe that the restriction of $\mathcal{M}$ to $D \times D$ is isomorphic to a direct sum of the modules $((\mathcal{O}D)b)v_t$ for all $t \in \mathcal{T}$. Here $(\mathcal{O}D)t \cong ((\mathcal{O}D)b)v_t$ in $\mathcal{O}(D \times D)$-mod and so $((\mathcal{O}D)b)v_t \cong \mathrm{Ind}^{\mathcal{O}(D\times D)}_{R_t}(\mathcal{O})$ in $\mathcal{O}(D \times D)$-mod, where

$R_t = \{(d, t^{-1}dt) \mid d \in D\}$ for each $t \in \mathcal{T}$. Thus [5, III, Lemma 4.6] implies that $\mathcal{M}$ has ΔD as a vertex and a trivial source.

Now we proceed to demonstrate that such a Morita equivalence can be chosen so as to satisfy (ii) of Theorem 2.

Let $\mathcal{I}((\mathcal{O}D)b) = \sum_{d \in D^{\#}} \mathcal{O}(d-1)b$ and $\mathcal{I}((\mathcal{O}Z(D))b) = \sum_{d \in Z(D)^{\#}} \mathcal{O}(d-1)b$ denote the augmentation ideals of $(\mathcal{O}D)b$ and $(\mathcal{O}Z(D))b$, resp. as in Lemma 1.14. Set $\mathcal{I}((\mathcal{O}L)b) = (\mathcal{O}L)b\mathcal{I}((\mathcal{O}Z(D))b)$ and $\mathcal{I}((\mathcal{O}K)b) = (\mathcal{O}K)b\mathcal{I}((\mathcal{O}D)b)$. Then $\mathcal{I}((\mathcal{O}L)b) = \mathcal{S}\mathcal{I}(\mathcal{O}Z(D)b) = \mathcal{I}(\mathcal{O}Z(D)b)\mathcal{S}$, $\mathcal{I}((\mathcal{O}L)b)$ is an ideal of $(\mathcal{O}L)b$, $\mathcal{I}((\mathcal{O}L)b) \le J((\mathcal{O}L)b)$, and $\mathcal{I}((\mathcal{O}L)b) = \oplus_{d \in Z(D)^{\#}}\mathcal{S}(d-1)$ and $(\mathcal{O}L)b = \mathcal{S} \oplus \mathcal{I}((\mathcal{O}L)b)$ in $\mathcal{S}$-mod-$\mathcal{S}$. Thus the $\mathcal{O}$-algebra projection $\pi_{\mathcal{S}} : (\mathcal{O}L)b = \mathcal{S} \oplus \mathcal{I}((\mathcal{O}L)b) \to \mathcal{S}$ in $\mathcal{S}$-mod-$\mathcal{S}$ is an epimorphism with $\mathrm{Ker}(\pi_{\mathcal{S}}) = \mathcal{I}((\mathcal{O}L)b)$. Moreover Proposition 2.1(c) implies that $C_{(\mathcal{O}L)b}(\mathcal{S}) = \mathcal{O}Z(D)b$.

Similar facts hold for $(\mathcal{O}K)b$: $\mathcal{I}((\mathcal{O}K)b) = \mathcal{S}\mathcal{I}((\mathcal{O}D)b) = \mathcal{I}((\mathcal{O}D)b)\mathcal{S}$, $\mathcal{I}((\mathcal{O}K)b)$ is an ideal of $(\mathcal{O}K)b$, $\mathcal{I}((\mathcal{O}K)b) \le J((\mathcal{O}K)b)$, $\mathcal{I}((\mathcal{O}K)b) = \oplus_{d \in D^{\#}}\mathcal{S}(d-1)b$ and $(\mathcal{O}K)b = \mathcal{S} \oplus \mathcal{I}((\mathcal{O}K)b)$ in $\mathcal{S}$-mod-$\mathcal{S}$. Also the projection $\pi_{\mathcal{S}} : (\mathcal{O}K)b = \mathcal{S} \oplus \mathcal{I}((\mathcal{O}K)b) \to \mathcal{S}$ in $\mathcal{S}$-mod-$\mathcal{S}$ is an epimorphism with $\mathrm{Ker}(\pi_{\mathcal{S}}) = \mathcal{I}((\mathcal{O}K)b)$ and $C_{(\mathcal{O}K)b}(\mathcal{S}) = (\mathcal{O}D)b$.

Similarly, we define $\mathcal{I}((\mathcal{O}D)W(b))$, $\mathcal{I}((\mathcal{O}Z(D))W(b))$, $\mathcal{I}((\mathcal{O}L_1)W(b))$, $\mathcal{I}((\mathcal{O}K_1)W(b))$, $\pi_{\mathcal{S}_1} : (\mathcal{O}L_1)W(b) \to \mathcal{S}_1$ and $\pi_{\mathcal{S}_1} : (\mathcal{O}K_1)W(b) \to \mathcal{S}_1$ and we have the corresponding facts.

Extending to $\mathcal{K}$, it is clear that

$$\theta = Tr_{\mathcal{S}} \circ \pi_{\mathcal{S}} : (\mathcal{O}K)b \to \mathcal{O} \text{ and}$$
$$\theta_1 = Tr_{\mathcal{S}_1} \circ \pi_{\mathcal{S}_1} : (\mathcal{O}K_1)W(b) \to \mathcal{O}$$

canonically induce irreducible characters of K and K_1, respectively, with D contained in their kernels and such that $\theta \in \mathrm{Irr}_{\mathcal{K}}(b)$ and $\theta_1 \in \mathrm{Irr}_{\mathcal{K}}(W(b))$.

Lemma 2.5. (a) $b + \mathcal{I}((\mathcal{O}L)b) \trianglelefteq ((\mathcal{O}L)b)^{\times}, ((\mathcal{O}L)b)^{\times} = \mathcal{S}^{\times}(b + \mathcal{I}((\mathcal{O}L)b)) = (b + \mathcal{I}((\mathcal{O}L)b))\mathcal{S}^{\times}$ *and* $\mathcal{S}^{\times} \cap (b + \mathcal{I}((\mathcal{O}L)b)) = b$*;*

(b)

$$N_{((\mathcal{O}L)b)^{\times}}(\mathcal{S}^{\times}) = \mathcal{S}^{\times} \times (b + \mathcal{I}(\mathcal{O}Z(D)b))$$

and

$$N_{b+\mathcal{I}((\mathcal{O}L)b)}(\mathcal{S}^{\times}) = C_{b+\mathcal{I}((\mathcal{O}L)b)}(\mathcal{S}^{\times}) = b + \mathcal{I}(\mathcal{O}Z(D)b).$$

Similar results hold for $(\mathcal{O}K)b$ *with* $(\mathcal{O}D)b$ *in place of* $\mathcal{O}Z(D)b$.

Proof. Clearly $b + \mathcal{I}((\mathcal{O}L)b) \subseteq b + J((\mathcal{O}L)b) \trianglelefteq ((\mathcal{O}L)b)^{\times}$. Let $i \in \mathcal{I}((\mathcal{O}L)b)$. Then there is an element $j \in J((\mathcal{O}L)b)$ such that $(b+i)(b+j) = b$. Thus $j = -i - ij \in \mathcal{I}((\mathcal{O}L)b)$ and hence $b + \mathcal{I}((\mathcal{O}L)b) \trianglelefteq ((\mathcal{O}L)b)^{\times}$. Let $x \in ((\mathcal{O}L)b)^{\times}$, so that $x = s + i$ for some $s \in \mathcal{S}$ and $i \in \mathcal{I}((\mathcal{O}L)b)$. The projection $\pi_{\mathcal{S}} : (\mathcal{O}L)b \to \mathcal{S}$ yields an element $t \in \mathcal{S}$ such that $st = ts = b$. Thus $x = stx = s(b + ti) \in \mathcal{S}^{\times}(b + \mathcal{I}((\mathcal{O}L)b))$ and, since $\pi_{\mathcal{S}}(x) = s$, (a) holds. For (b), note that $\mathcal{S}^{\times} \trianglelefteq N_{((\mathcal{O}L)b)^{\times}}(\mathcal{S}^{\times}) = \mathcal{S}^{\times} N_{b+\mathcal{I}((\mathcal{O}L)b)}(\mathcal{S}^{\times})$ and $N_{b+\mathcal{I}((\mathcal{O}L)b)}(\mathcal{S}^{\times}) \trianglelefteq N_{((\mathcal{O}L)b)^{\times}}(\mathcal{S}^{\times})$. Thus

$$[\mathcal{S}^{\times}, N_{b+\mathcal{I}(\mathcal{O}L)b)}(\mathcal{S}^{\times})] \subseteq \mathcal{S}^{\times} \cap (b + \mathcal{I}((\mathcal{O}L)b)) = b$$

and (b) follows from Propositions 2.1(c) and 1.12(e).

Corollary 2.6. $b + J((\mathcal{O}L)b) = (b + J(\mathcal{S}))(b + \mathcal{I}((\mathcal{O}L)b)) = (b + \mathcal{I}((\mathcal{O}L)b))(b + J(\mathcal{S}))$. *A similar result holds for* $(\mathcal{O}K)b$.

Proof. Since $b+J((\mathcal{O}L)b) = ((b+J((\mathcal{O}L)b))\cap\mathcal{S}^\times)(b+\mathcal{I}((\mathcal{O}L)b))$ by Lemma 2.5(a) and $(b+J((\mathcal{O}L)b))\cap\mathcal{S}^\times = b+J(\mathcal{S})$ by Proposition 1.12(a), we are done.

Lemma 2.7. (a) *If $x \in ((\mathcal{O}G)b)^\times$, acting by conjugation, normalizes $(\mathcal{O}L)b$ and $\mathcal{I}((\mathcal{O}L)b)$, and if $u \in (\mathcal{O}L)b$, then ${}^x\pi_{\mathcal{S}}(u) = \pi_{{}^x\mathcal{S}}({}^xu)$; and*

(b) *if $x \in b+\mathcal{I}((\mathcal{O}L)b)$ and $u \in (\mathcal{O}L)b$, then $x^{-1} \in b+\mathcal{I}((\mathcal{O}L)b)$ and $\pi_{\mathcal{S}}({}^xu) = \pi_{\mathcal{S}}(u)$. Similar results hold for $(\mathcal{O}K)b$.*

Proof. Assume the conditions of (a) so that $u = s+i$ for unique $s \in \mathcal{S}$ and $i \in \mathcal{I}((\mathcal{O}L)b)$. Then ${}^xu = {}^xs + {}^xi$, so that $\pi_{{}^x\mathcal{S}}({}^xu) = {}^xs = {}^x\pi_{\mathcal{S}}(u)$. Assume the conditions of (b) so that, $x^{-1} \in b+\mathcal{I}((\mathcal{O}L)b)$ by Lemma 2.5(a) and $\pi_{\mathcal{S}}({}^xu) = \pi_{\mathcal{S}}(xux^{-1}) = \pi_{\mathcal{S}}(x)\pi_{\mathcal{S}}(u)\pi_{\mathcal{S}}(x^{-1}) = \pi_{\mathcal{S}}(u)$ and we are done.

Remark 2.8. If $g \in G$, then every element of $((\mathcal{O}L)b)^\times(gb)$ normalizes both $(\mathcal{O}L)b$ and $\mathcal{I}((\mathcal{O}L)b)$.

Let Σ denote the set of maximal $\mathcal{O}$-semi-simple $\mathcal{O}$-subalgebras of $(\mathcal{O}L)b$. Here $\mathcal{S} \in \Sigma$, all elements of Σ are $\mathcal{O}$-simple and $(\mathcal{O}L)b = \mathcal{S}^\times(b+\mathcal{I}((\mathcal{O}L)b))$. Thus $b+\mathcal{I}((\mathcal{O}L)b)$ acts transitively on Σ by conjugation ([18, Lemma 45.6]).

Lemma 2.9. *Let X be a finite subgroup of $Aut_{\mathcal{O}}((\mathcal{O}L)b)$ of order prime to p that leaves $\mathcal{I}((\mathcal{O}L)b)$ invariant. Then there is an $x \in b+\mathcal{I}((\mathcal{O}L)b)$ such that ${}^x\mathcal{S}$ is X-invariant.*

Proof. Set $\mathcal{G} = (b+J((\mathcal{O}L)b))\rtimes X$, so that $\mathcal{G}$ permutes Σ. Since $b+\mathcal{I}((\mathcal{O}L)b)$ and $b+J((\mathcal{O}L)b)$ are transitive on Σ, $\mathcal{G} = (b+J((\mathcal{O}L)b))N_{\mathcal{G}}(\mathcal{S}^\times) = (b+\mathcal{I}((\mathcal{O}L)b))N_{\mathcal{G}}(\mathcal{S}^\times)$ since $b+\mathcal{I}((\mathcal{O}L)b) \trianglelefteq \mathcal{G}$ by Lemma 2.5. Applying [18, Lemma 45.6], it suffices to prove that $N_{b+J((\mathcal{O}L)b)}(\mathcal{S}^\times)$ has a complement in $N_{\mathcal{G}}(\mathcal{S}^\times)$. Here $N_{b+J((\mathcal{O}L)b)}(\mathcal{S}^\times) = N_{b+J((\mathcal{O}L)b)}(\mathcal{S}) = (b+\mathcal{I}(\mathcal{O}Z(D)b)) \times (b+J(\mathcal{S}))$ by Lemma 2.5(b) and Lemma 1.5(b). Also $b+J(\mathcal{S})$ and $b+\mathcal{I}(\mathcal{O}Z(D)b) = N_{b+\mathcal{I}((\mathcal{O}L)b)}(\mathcal{S}^\times)$ are normal subgroups of $N_{\mathcal{G}}(\mathcal{S}^\times)$ and $N_{\mathcal{G}}(\mathcal{S}^\times)/N_{b+J((\mathcal{O}L)b)}(\mathcal{S}^\times) \tilde{=} X$. Set

$$\overline{N_{\mathcal{G}}(\mathcal{S}^\times)} = N_{\mathcal{G}}(\mathcal{S}^\times)/(b+\mathcal{I}(\mathcal{O}Z(D)b)).$$

Then $b+J(\mathcal{S}) \tilde{=} \overline{b+J(\mathcal{S})} \trianglelefteq \overline{N_{\mathcal{G}}(\mathcal{S}^\times)}$ and [18, Lemma 45.6] implies that $\overline{b+J(\mathcal{S})}$ has a complement $\bar{Y}$ in $\overline{N_{\mathcal{G}}(\mathcal{S}^\times)}$. The inverse image Y of $\bar{Y}$ in $N_{\mathcal{G}}(\mathcal{S}^\times)$ satisfies $b+\mathcal{I}(\mathcal{O}Z(D)b) \leq Y$ and $Y/(b+\mathcal{I}(\mathcal{O}Z(D)b)) \tilde{=} X$. Since $b+\mathcal{I}(\mathcal{O}Z(D)b)$ is an Abelian group, [9, I, Hauptsatz 17.4] and Lemma 1.14(b) imply that $b+\mathcal{I}(\mathcal{O}Z(D)b)$ has a complement B in Y. Clearly B is a complement to $N_{b+J((\mathcal{O}L)b)}(\mathcal{S}^\times)$ in $N_{\mathcal{G}}(\mathcal{S}^\times)$ and we are done.

Corollary 2.10. *A leaves invariant an element of Σ.*

We shall henceforth assume that $\mathcal{S} \in \Sigma$ is A-invariant. It follows that $\pi_{\mathcal{S}} : (\mathcal{O}L)b \to \mathcal{S}$ and $\pi_{\mathcal{S}} : (\mathcal{O}K)b \to \mathcal{S}$ are A-projections and that $C_{(\mathcal{O}H)b}(\mathcal{S}) = \oplus_{t\in\mathcal{T}}(\mathcal{O}Z(D)b)v_t$ and $C_{(\mathcal{O}G)b}(\mathcal{S}) = \oplus_{t\in\mathcal{T}}((\mathcal{O}D)b)v_t$ are A-invariant. In fact, for each $t \in \mathcal{T}$, $C_{((\mathcal{O}L)b)(tb)}(\mathcal{S}) = C_{((\mathcal{O}L)b)v_t}(\mathcal{S}) = \mathcal{O}Z(D)bv_t$ and $C_{(\mathcal{O}G)b(tb)}(\mathcal{S}) = (\mathcal{O}D)bv_t$ are A-invariant and, from Proposition 2.1(e), $v_t = w_t(tb)$ for some $w_t \in ((\mathcal{O}L)b)^\times$. Consequently $w_t = s_t\alpha_t$ for unique $s_t \in \mathcal{S}^\times$ and $\alpha_t \in b+\mathcal{I}((\mathcal{O}L)b)$ by Lemma 2.5(a) for each $t \in \mathcal{T}$. Since $w_1 = b$, we have $s_1 = \alpha_1 = b$.

Lemma 2.11. (a) *A acts trivially on $C_{(\mathcal{O}H)b}(\mathcal{S}) = \oplus_{t\in\mathcal{T}}\mathcal{O}Z(D)bv_t$ and $C_{(\mathcal{O}G)b}(\mathcal{S}) = \oplus_{t\in\mathcal{T}}(\mathcal{O}D)bv_t$; and*

(b) *s_t and α_t are fixed by A for all $t \in \mathcal{T}$.*

Proof. Let $t \in \mathcal{T}$ and $a \in A$. Then $v_t^a = \alpha v_t$ for a unique $\alpha \in \mathcal{O}Z(D)b$. Hence $\alpha^{|a|} = b$. Since $(|a|, p) = 1$, Lemma 1.13(a) and Proposition 1.8 imply that $\alpha = \gamma b$ for a unique $\gamma \in \mathcal{O}^\times$ such that $\gamma^{|a|} = 1$. However $v_t^m = \delta b$ for a unique $\delta \in \Omega \leq \{x \in \mathcal{O}^\times | x^m = 1\}$ by Proposition 2.1. Hence $(v_t^a)^m = \gamma^m(\delta b) = (\delta b)^a = \delta b$ so that $\gamma^m = 1$. As $(|a|, m) = 1$, we conclude that $\gamma = 1$ and (a) follows. Since $v_t = s_t\alpha_t(tb)$ for unique $s_t \in \mathcal{S}^\times$ and $\alpha_t \in b + \mathcal{I}((\mathcal{O}L)b)$ and both $\mathcal{S}^\times$ and $b + \mathcal{I}((\mathcal{O}L)b)$ are A-invariant, Lemma 2.5(a) implies (b). Our proof is complete.

Recall that $G = \bigcup_{t\in\mathcal{T}} Kt$, where the union is disjoint. We define $\pi_{\mathcal{S}}^* : G \to \mathcal{S}^\times$ by $kt \mapsto \pi_{\mathcal{S}}(kb)s_t^{-1}$ for all $k \in K$ and all $t \in \mathcal{T}$ and we extend $\pi_{\mathcal{S}}^*$ to an $\mathcal{O}$-linear map $\pi_{\mathcal{S}}^* : \mathcal{O}G \to \mathcal{S}$. Since $s_1 = b$, $\pi_{\mathcal{S}}^*$ extends the $\mathcal{O}$-algebra A-epimorphism $\pi_{\mathcal{S}} : (\mathcal{O}K)b \to \mathcal{S}$. Clearly $\pi_{\mathcal{S}}^*(\alpha) = \pi_{\mathcal{S}}^*(\alpha b)$ for all $\alpha \in \mathcal{O}G$.

Lemma 2.12. *Let $\alpha, \beta \in \mathcal{O}K$ and $t \in \mathcal{T}$. Then:*

(a) $\pi_{\mathcal{S}}^*(\alpha)\pi_{\mathcal{S}}^*(\beta t) = \pi_{\mathcal{S}}((\alpha\beta)b)s_t^{-1} = \pi_{\mathcal{S}}^*(\alpha\beta t)$*;*

(b) $\pi_{\mathcal{S}}^*(\alpha t\beta) = \pi_{\mathcal{S}}^*(\alpha({}^t\beta)t) = \pi_{\mathcal{S}}(\alpha({}^t\beta)b)s_t^{-1} = \pi_{\mathcal{S}}^*(\alpha)\pi_{\mathcal{S}}^*(({}^t\beta)t) = \pi_{\mathcal{S}}^*(\alpha)\pi_{\mathcal{S}}^*(t\beta)$*;*

(c) $\pi_{\mathcal{S}}^*(1) = b = \pi_{\mathcal{S}}^*(b)$, $\pi_{\mathcal{S}}^*((1-b)\mathcal{O}G) = \pi_{\mathcal{S}}^*(\mathcal{O}G(1-b)) = 0$ *and* $\pi_{\mathcal{S}}^* : (\mathcal{O}G)b \to \mathcal{S}$ *is a surjective $\mathcal{O}$-linear map; and*

(d) $\pi_{\mathcal{S}}^*(v_t) = b$ *and* $s_t^{-1}\pi_{\mathcal{S}}(\beta b)s_t = \pi_{\mathcal{S}}({}^t(\beta b))$.

Proof. Clearly (a)–(c) hold. Since $v_t = s_t\alpha_t(tb)$, we have $\pi_{\mathcal{S}}^*(v_t) = \pi_{\mathcal{S}}(s_t\alpha_t)s_t^{-1} = s_ts_t^{-1} = b$. Also $s_t^{-1}v_t = \alpha_t(tb) \in N_{((\mathcal{O}G)b)^\times}(\mathcal{S})$. Thus

$$s_t^{-1}\pi_{\mathcal{S}}(\beta b)s_t = s_t^{-1}v_t\pi_{\mathcal{S}}(\beta b)v_t^{-1}s_t = \pi_{\mathcal{S}}({}^{\alpha_t(tb)}(\beta b)) = \pi_{\mathcal{S}}({}^{tb}(\beta b)) = \pi_{\mathcal{S}}({}^t(\beta b))$$

using Lemma 2.7, the fact that α_t and tb normalize $(\mathcal{O}L)b$ and $\mathcal{I}((\mathcal{O}L)b)$ and the fact that $\alpha_t \in \operatorname{Ker}(\pi_{\mathcal{S}})$. Our proof is complete.

Lemma 2.13. *Let $k_1, k_2 \in K$ and $t_1, t_2 \in \mathcal{T}$. Here $t_1t_2 = \ell t_3$ and $t_1^{-1} = \ell' t_4$ for unique $\ell, \ell' \in L_1$ and $t_3, t_4 \in \mathcal{T}$. Then:*

(a) $\pi_{\mathcal{S}}^*(k_1t_1)\pi_{\mathcal{S}}^*(k_2t_2) = c(t_1L_1, t_2L_1)^{-1}\pi_{\mathcal{S}}^*(k_1t_1k_2t_2)$*;*

(b) $\pi_{\mathcal{S}}^*(k_1t_1)^{-1} = c(t_1L_1, t_4L_1)\pi_{\mathcal{S}}^*((k_1t_1)^{-1})$*; and*

(c) $\pi_{\mathcal{S}}^*((k_1t_1)^a) = \pi_{\mathcal{S}}^*(k_1^at_1) = \pi_{\mathcal{S}}^*(k_1^a)s_{t_1}^{-1} = \pi_{\mathcal{S}}^*(k_1t_1)^a$ *for all $a \in A$ and $\pi_{\mathcal{S}}^* : \mathcal{O}G \to \mathcal{S}$ is an A-epimorphism in $\mathcal{O}$-mod.*

Proof. Clearly $(k_1t_1)(k_2t_2) = k_1({}^{t_1}k_2)\ell t_3$ and $\pi_{\mathcal{S}}^*(k_1t_1k_2t_2) = \pi_{\mathcal{S}}(k_1({}^{t_1}k_2)\ell b)s_{t_3}^{-1}$. Also $\pi_{\mathcal{S}}^*(k_1t_1)\pi_{\mathcal{S}}^*(k_2t_2) = \pi_{\mathcal{S}}(k_1b)s_{t_1}^{-1}\pi_{\mathcal{S}}(k_2b)s_{t_2}^{-1} = \pi_{\mathcal{S}}(k_1b)\pi_{\mathcal{S}}(({}^{t_1}k_2)b)s_{t_1}^{-1}s_{t_2}^{-1} = \pi_{\mathcal{S}}(k_1({}^{t_1}k_2)b)s_{t_1}^{-1}s_{t_2}^{-1}$ using Lemma 2.12(d). Since $v_{t_1}v_{t_2} = c(t_1L_1, t_2L_2)v_{t_3}$, we have $s_{t_1}^{-1}v_{t_1}s_{t_2}^{-1}v_{t_2} = s_{t_1}^{-1}s_{t_2}^{-1}c(t_1L_1, t_2L_1)v_{t_3} = \alpha_{t_1}(t_1b)\alpha_{t_2}(t_2b) = \alpha_{t_1}({}^{t_1}\alpha_{t_2})(\ell t_3)b = \alpha_{t_1}({}^{t_1}\alpha_{t_2})(\ell b)\alpha_{t_3}^{-1}s_{t_3}^{-1}v_{t_3}$. Thus, $c(t_1L_1, t_2L_1)s_{t_1}^{-1}s_{t_2}^{-1} = \alpha_{t_1}({}^{t_1}\alpha_{t_2})(\ell b)\alpha_{t_3}^{-1}s_{t_3}^{-1}$ and applying $\pi_{\mathcal{S}}$, we conclude that $c(t_1L_1, t_2L_1)s_{t_1}^{-1}s_{t_2}^{-1} = \pi_{\mathcal{S}}(\ell b)s_{t_3}^{-1}$ and (a) holds. Then (b) and (c) are immediate and we are done.

Recall the $\mathcal{O}$-algebra isomorphism $\Phi : C_{(\mathcal{O}C)W(b)}(\mathcal{S}_1) = \oplus_{t\in\mathcal{T}}((\mathcal{O}D)W(b)v_t') \to C_{(\mathcal{O}G)b}(\mathcal{S}) = \oplus_{t\in\mathcal{T}}(\mathcal{O}D)bv_t$ such that $(\alpha W(b))v_t' \mapsto (\alpha b)v_t$ for all $\alpha \in \mathcal{O}D$ and all $t \in \mathcal{T}$. Thus $v_{t_1}'v_{t_2}' = c(t_1L_1, t_2L_1)v_{t_3}'$, where $t_1t_2 \in L_1t_3$. For $t \in \mathcal{T}$, $v_t' = w_t'(tW(b))$, where $w_t' \in ((\mathcal{O}L_1)W(b))^\times = \mathcal{S}_1^\times(W(b) + \mathcal{I}((\mathcal{O}L_1)W(b)))$, so that $w_t' = s_t'\alpha_t'$ for unique $s_t' \in \mathcal{S}_1^\times$ and $\alpha_t' \in W(b) + \mathcal{I}((\mathcal{O}L_1)W(b))$. As above, define $\pi_{\mathcal{S}_1}^* : C = \bigcup_{t\in\mathcal{T}} K_1t \to \mathcal{S}_1^\times$ by $k_1t \mapsto \pi_{\mathcal{S}_1}(k_1W(b))(s_t')^{-1}$ for all $k_1 \in K_1$ and $t \in \mathcal{T}$ and extend $\pi_{\mathcal{S}_1}^*$ to an $\mathcal{O}$-linear map $\pi_{\mathcal{S}_1}^* : \mathcal{O}C \to \mathcal{S}_1$. The proof of Lemma 2.13 with $C, K_1, \mathcal{S}_1$ in place of $G, K, \mathcal{S}$ yields the following lemma.

Lemma 2.14. *Let $k_1, k_2 \in K_1$ and $t_1, t_2 \in \mathcal{T}$. Here $t_1t_2 = \ell t_3$ and $t_1^{-1} = \ell' t_4$ for unique $\ell, \ell' \in L_1$ and $t_3, t_4 \in \mathcal{T}$. Then:*

(a) $\pi^*_{\mathcal{S}_1}(k_1t_1)\pi^*_{\mathcal{S}_1}(k_2t_2) = c(t_1L_1, t_2L_1)^{-1}\pi^*_{\mathcal{S}_1}(k_1t_1k_2t_2)$;

(b) $\pi^*_{\mathcal{S}_1}(k_1t_1)^{-1} = c(t_1L_1, t_4L_1)\pi^*_{\mathcal{S}_1}((k_1t_1)^{-1})$; *and*

(c) $\pi^*_{\mathcal{S}_1} : \mathcal{O}C \to \mathcal{S}_1$ *is an epimorphism of $\mathcal{O}$-modules.*

Observe that $(\mathcal{O}H)b$ can be viewed as an H/L-crossed product $\mathcal{O}$-algebra with $((\mathcal{O}H)b)_{tL} = (\mathcal{O}L)b(tb)$ for all $t \in \mathcal{T}$.

Thus Lemma 1.6 yields:

Lemma 2.15. $((\mathcal{O}H)b)^\times \cap ((\mathcal{O}L)b) = ((\mathcal{O}L)b)^\times$.

Lemma 2.16. *Let $t \in \mathcal{T}$. Then:*

(a) *if $y \in (Kt)_{p'}$, then y is D-conjugate to an element $u \in (Lt)_{p'}$; and*

(b) *if $u \in (Lt)_{p'}$, then $C_D(u) = C_D(t)$, $ub \in (\mathcal{O}L)b(tb) = (\mathcal{O}L)v_t$ and there is an element $\gamma \in b + \mathcal{I}((\mathcal{O}L)b)$ such that $(ub)^\gamma = sv_t$, where $s \in \mathcal{S}^\times$. Moreover, if $d \in C_D(u)$, then $(udb)^\gamma = s((db)v_t)$ and $\pi^*_{\mathcal{S}}(udb) = s$.*

Proof. If $y \in (Kt)_{p'}$, then, as $G = HD$, y is D-conjugate to an element $u \in (H \cap (Kt))_{p'} = (Lt)_{p'}$ and (a) holds. Let $u \in (Lt)_{p'}$. Then $C_D(u) = C_D(t)$ and $ub \in ((\mathcal{O}L)b)v_t$. Set $X = \langle u \rangle$. Then Lemma 2.9 yields an element $z \in b + \mathcal{I}((\mathcal{O}L)b)$ such that X leaves ${}^z\mathcal{S}$ invariant. Consequently $(ub)^z \in N_{((\mathcal{O}H)b)^\times}(\mathcal{S}) \cap ((\mathcal{O}L)bv_t)$. Note that $((\mathcal{O}H)b)^\times \cap ((\mathcal{O}L)bv_t) = (\mathcal{O}L)b)^\times v_t$ because of Lemma 1.6. Thus $(ub)^z \in (N_{((\mathcal{O}H)b)^\times}(\mathcal{S}) \cap ((\mathcal{O}L)b)^\times)v_t = N_{((\mathcal{O}L)b)^\times}(\mathcal{S})v_t = (\mathcal{S}^\times \times (b + \mathcal{I}(\mathcal{O}Z(D)b)))v_t$ because of Lemma 2.5. Thus $(ub)^z = s\alpha v_t$ for unique $s \in \mathcal{S}^\times$ and $\alpha \in b + \mathcal{I}(\mathcal{O}Z(D)b)$. Set $f = |u|$, so that $(f, p) = 1$ and $s^f(\alpha v_t)^f = b$. Consequently, $s^f \in \mathcal{S}^\times \cap C_{(\mathcal{O}G)b}(\mathcal{S}) = \mathcal{O}^\times b$ and $s^f = \sigma b$ for a unique $\sigma \in \mathcal{O}^\times$. Since $(f, p) = 1$, there is an element $\delta \in \mathcal{O}^\times$ such that $\delta^f = \sigma$. Then $(ub)^z = (\delta^{-1}s)(\alpha(\delta v_t))$, where $(\delta^{-1}s)^f = b = (\alpha(\delta v_t))^f$. Here $\delta^f v_t^f = (\delta v_t)^f \in (b + \mathcal{O}Z(D)b) \cap (\oplus_{t\in\mathcal{T}}\mathcal{O}v_t) = b$. Since $\alpha(\delta v_t) \in (b + J(\mathcal{O}Z(D)b))(\delta v_t) \subseteq (\mathcal{O}Z(D)b)^\times(\delta v_t)$ and $v_t\alpha v_t^{-1} = t\alpha t^{-1}$ for all $\alpha \in (\mathcal{O}D)b$ by Proposition 2.1(e), Lemma 1.11 with $X = (b + J(\mathcal{O}Z(D)b))E$ where $e = \delta v_t$ and $E = \langle \delta v_t \rangle$ and Lemma 1.14(a) imply the existence of an element $x \in b + \mathcal{I}(\mathcal{O}Z(D)b)$ such that $(\alpha(\delta v_t))^x = \delta v_t$. Thus $\gamma = zx \in (b + \mathcal{I}((\mathcal{O}L)b))$ is such that $(ub)^\gamma = (\delta^{-1}s)(\alpha(\delta v_t))^x = (\delta^{-1}s)(\delta v_t) = sv_t$. If $d \in C_D(u) = C_D(t)$, then $((ud)b)^\gamma = (ub)^\gamma(db) = s((db)v_t)$. Let $\gamma = b + j$ and $\gamma^{-1} = b + j'$, where $j, j' \in \mathcal{I}((\mathcal{O}L)b)$. Then $(b+j')((ud)b)(b+j) = (b+j')(b+{}^uj)(ud)b$, where $(b+j')(b+{}^uj) \in b+\mathcal{I}((\mathcal{O}L)b)$, so Lemma 2.12 implies that $\pi^*_{\mathcal{S}}(((ud)b)^\gamma) = \pi^*_{\mathcal{S}}((b+j')(ud)b(b+j)) = \pi^*_{\mathcal{S}}((ud)b) = \pi^*_{\mathcal{S}}(s(db)v_t) = s$ and we are done.

We extend $\theta = Tr_{\mathcal{S}} \circ \pi_{\mathcal{S}} : (\mathcal{O}K)b \to \mathcal{O}$ and $\theta_1 = Tr_{\mathcal{S}_1} \circ \pi_{\mathcal{S}_1} : (\mathcal{O}K_1)W(b) \to \mathcal{O}$ to $\mathcal{O}$-linear maps $\theta = Tr_{\mathcal{S}} \circ \pi^*_{\mathcal{S}} : \mathcal{O}G \to \mathcal{O}$ such that $g \mapsto Tr_{\mathcal{S}}(\pi^*_{\mathcal{S}}(gb))$ for all $g \in G$ and $\theta_1 = Tr_{\mathcal{S}_1} \circ \pi^*_{\mathcal{S}_1} : \mathcal{O}C \to \mathcal{O}$ such that $g \mapsto Tr_{\mathcal{S}_1}(\pi^*_{\mathcal{S}_1}(gW(b)))$ for all $g \in C$. Set $\mathcal{B} = C_{(\mathcal{O}G)b}(\mathcal{S})$ and $\mathcal{B}' = C_{\mathcal{O}CW(b)}(\mathcal{S}_1)$.

For the remainder of this article we extend coefficients to $\mathcal{K}$ so that $\mathcal{K}\mathcal{S} \cong M_r(\mathcal{K})$, $\mathcal{K}\mathcal{S}_1 \cong M_{r_1}(\mathcal{K})$, $\mathcal{K}C_{(\mathcal{O}G)b}(\mathcal{S}) = \mathcal{K}\mathcal{B} = \oplus_{\substack{t\in\mathcal{T}\\ d\in D}}(\mathcal{K}(db)v_t)$, $\mathcal{K}C_{(\mathcal{O}C)W(b)}(\mathcal{S}_1) = \mathcal{K}\mathcal{B}' = \oplus_{\substack{t\in\mathcal{T}\\ d\in D}}(\mathcal{K}(dW(b))v'_t)$, the multiplication maps

$$\mu : (\mathcal{K}\mathcal{S}) \otimes_{\mathcal{K}} (\mathcal{K}\mathcal{B}) \to (\mathcal{K}G)b,$$

$$\mu_1 : (\mathcal{K}\mathcal{S}_1) \otimes_{\mathcal{K}} (\mathcal{K}\mathcal{B}') \to (\mathcal{K}C)W(b)$$

are $\mathcal{K}$-algebra isomorphisms, etc.

Clearly

$$\mathrm{Irr}_{\mathcal{K}}((\mathcal{K}\mathcal{S})\otimes_{\mathcal{K}}(\mathcal{K}\mathcal{B}))=\{(Tr_{\mathcal{S}}*\psi)\mid\psi\in\mathrm{Irr}_{\mathcal{K}}(\mathcal{K}\mathcal{B})\},$$

where $(Tr_{\mathcal{S}}*\psi)(s\otimes_{\mathcal{K}}\beta)=Tr_{\mathcal{S}}(s)\psi(\beta)$ for all $s\in\mathcal{S}$ and $\beta\in\mathcal{B}$ and for all $\psi\in\mathrm{Irr}_{\mathcal{K}}(\mathcal{K}\mathcal{B})$. Thus $\mathrm{Irr}_{\mathcal{K}}(b)=\{(Tr_{\mathcal{S}}*\psi)\mu^{-1}\mid\psi\in\mathrm{Irr}_{\mathcal{K}}(\mathcal{K}\mathcal{B})\}$. Set $\theta_\psi=(Tr_{\mathcal{S}}*\psi)\mu^{-1}$ for all $\psi\in\mathrm{Irr}_{\mathcal{K}}(\mathcal{K}\mathcal{B})$. We view $\mathrm{Irr}_{\mathcal{K}}(\mathcal{K}B)$ as a subset of $\mathrm{Irr}_{\mathcal{K}}(\mathcal{K}G)$ in the canonical fashion.

Fix $\psi\in\mathrm{Irr}_{\mathcal{K}}(\mathcal{K}\mathcal{B})$ and $t\in\mathcal{T}$ and let $g\in Kt$. Since D is the defect group of b, $\theta_\psi(g)=0$ if $g_p\notin D$ by [5, IV, Lemma 2.4]. Hence, by Lemma 1.22, we have

$$\theta_\psi(g)=0 \text{ if } gD\notin(G/D)_{p'}. \tag{2.3}$$

Suppose that $gD\in(G/D)_{p'}$. Then, by Lemma 1.22, $g_p\in D$ and $g_{p'}\in Kt$. Since $G=HD$, $g_{p'}$ is D-conjugate to an element $u\in H\cap(Kt)=Lt$ and $uD=gD\subseteq Kt$. By Lemma 2.16, there is an element $\sigma\in b+\mathcal{I}((\mathcal{O}L)b)$ such that $(ub)^\sigma=sv_t$, where $s\in\mathcal{S}$ and $\pi^*_{\mathcal{S}}(gb)=\pi^*_{\mathcal{S}}(ub)=s$. Also $C_D(u)b=C_D(t)b=C_{Db}(v_t)$ since $u\in Lt$ and we have $gD=uD=\bigcup_{f\in\mathcal{F}}(uC_D(u))^f$, where $\mathcal{F}$ is a right transversal of $C_D(u)$ in D and the union is disjoint. Moreover if $d\in C_D(u)$, then $(udb)^\sigma=s((db)v_t)$ since $\sigma\in b+\mathcal{I}((\mathcal{O}L)b)$. Thus:

$$\begin{aligned}&\text{if } d\in C_D(u),\ \text{then } \theta_\psi(ud)=Tr_{\mathcal{S}}(s)\psi((db)v_t),\ \text{where } \pi^*_{\mathcal{S}}((ud)b)=s;\ \text{and}\\ &\sum_{d\in D}\theta_\psi(ud)\theta_\psi((ud)^{-1})=|D:C_D(t)\mid Tr_{\mathcal{S}}(s)Tr_{\mathcal{S}}(s^{-1})\\ &\qquad\qquad\cdot\sum_{d\in C_D(t)}(\psi((db)v_t)\psi((db)v_t)^{-1})).\end{aligned}\tag{2.4}$$

Remark 2.17. Clearly (2.3) and (2.4) reduce to [16, Theorem 7] (cf. [5, V, Theorem 4.7]) when $G=K$ and present a description of $\mathrm{Irr}_{\mathcal{K}}(b)$ that differs from the description of $\mathrm{Irr}_{\mathcal{K}}(b)$ given in [16, Section 3] and that is consonant with [16, Theorems 5 and 6] and with [15, Theorem A].

Utilizing $\Phi:\mathcal{K}\mathcal{B}'\to\mathcal{K}\mathcal{B}$ (the canonic extension of the isomorphism $\Phi:\mathcal{B}'\to\mathcal{B}$), we have $\mathrm{Irr}_{\mathcal{K}}(\mathcal{K}\mathcal{B}')=\{\psi\Phi|\psi\in\mathrm{Irr}_{\mathcal{K}}(\mathcal{K}\mathcal{B})\}$.

Clearly $\mathrm{Irr}_{\mathcal{K}}(\mathcal{K}\mathcal{S}_1\otimes_{\mathcal{K}}\mathcal{K}\mathcal{B}')=\{Tr_{\mathcal{S}_1}*(\psi\Phi)\mid\psi\in\mathrm{Irr}_{\mathcal{K}}(\mathcal{K}\mathcal{B})\}$, where $Tr_{\mathcal{S}_1}*(\psi\Phi)$ is as defined above for all $\psi\in\mathrm{Irr}_{\mathcal{K}}(\mathcal{K}\mathcal{B})$. Also $\mathrm{Irr}_{\mathcal{K}}(W(b))=\{(Tr_{\mathcal{S}_1}*(\psi\Phi))\mu_1^{-1}\mid\psi\in\mathrm{Irr}_{\mathcal{K}}(\mathcal{K}\mathcal{B})\}$ and we set $\theta_{1\psi}=(Tr_{\mathcal{S}_1}*(\psi\Phi))\mu_1^{-1}$ for all $\psi\in\mathrm{Irr}_{\mathcal{K}}(\mathcal{K}\mathcal{B})$. Clearly the Morita equivalence above induces the bijection of $\mathrm{Irr}_{\mathcal{K}}(b)$ onto $\mathrm{Irr}_{\mathcal{K}}(W(b))$ that sends $\theta_\psi\mapsto\theta_{1\psi}$ for all $\psi\in\mathrm{Irr}_{\mathcal{K}}(\mathcal{K}\mathcal{B})$.

Note that $C_{G/D}(A)=C/D$ and let $g\in K_1t$. Then, as above,

$$\theta_{1\psi}(g)=0 \text{ if } gD\notin(C/D)_{p'}. \tag{2.5}$$

Suppose that $gD\in(C/D)_{p'}$. Then, as above, $gD=uD$ for some $u\in(L_1t)\cap C_{p'}$ and $uD=\bigcup_{f\in\mathcal{F}}(uC_D(u))^f$, where $\mathcal{F}$ is a right transversal of $C_D(u)$ in D and the union is disjoint. Set $s_1=\pi^*_{\mathcal{S}_1}(u)$, so that $\theta_{1\psi}(ud)=Tr_{\mathcal{S}_1}(s_1)\psi((db)v_t)$ for all $d\in C_D(u)$. Consequently,

$$\begin{aligned}&\sum_{d\in D}\theta_{1\psi}(ud)\theta_{1\psi}((ud)^{-1})=|D:C_D(t)|Tr_{\mathcal{S}_1}(s_1)Tr_{\mathcal{S}_1}(s_1^{-1})\\ &\qquad\cdot\sum_{d\in C_D(t)}(\psi((db)v_t)\psi(((db)v_t)^{-1}))\end{aligned}\tag{2.6}$$

and

$$\begin{aligned}\sum_{d\in D}\theta_\psi(ud)\theta_{1\psi}((ud)^{-1}) &= |D:C_D(t)|(Tr_{\mathcal{S}}(s))(Tr_{\mathcal{S}_1}(s_1^{-1}))\\ &\cdot\sum_{d\in C_D(t)}(\psi((db)v_t)\psi(((db)v_t)^{-1})),\end{aligned}\tag{2.7}$$

where $\pi^*_{\mathcal{S}}(ud) = s$ and $\pi^*_{\mathcal{S}_1}((ud)) = s_1$, for all $d \in D$.

Recall that $\Omega = \langle c(t_1L_1, t_2L_1)|t_1, t_2 \in \mathcal{T}\rangle$ is a subgroup of $\mathcal{O}^\times$ of order n dividing m so that $(n, p) = 1$. Let $\gamma : \Omega \to \mathcal{O}^\times$ denote the inclusion linear character (such that $\omega \mapsto \omega$ for all $\omega \in \Omega$).

We inflate $c \in Z^2(H_1/L_1, \mathcal{O}^\times)$ to an element $\hat{c} \in Z^2(G, \mathcal{O}^\times)$, where $\hat{c}(k_1t_1, k_2t_2) = c(t_1L_1, t_2L_1)$ for all $t_1, t_1 \in \mathcal{T}$ and all $k_1, k_1 \in K$.

Using $\hat{c}^{-1} \in Z^2(G, \mathcal{O}^\times)$, we construct the group $\hat{G} = \Omega\tilde{\times}G$ as in Lemma 1.2(d), where $|\hat{G}| = n|G|$. Note that

$$(\omega_1, k)(\omega_2, g) = (\omega_1\omega_2, kg), \qquad (\omega_2, g)(\omega_1, k) = (\omega_1\omega_2, gk)$$

and

$$(\omega_2, g)(\omega_1, k)(\omega_2, g)^{-1} = (\omega_1, gkg^{-1})$$

for all $\omega_1, \omega_2 \in \Omega$, all $g \in G$ and all $k \in K$ since $\hat{c}(g, h) = 1$ if $g \in K$ or $h \in K$. Thus $\hat{K} = \{(1, k)|k \in K\}$, $\hat{L} = \{(1, \ell)|\ell \in L\}$ and $\hat{D} = \{(1, d)|d \in D\}$ are normal subgroups of $\hat{G}$ and are isomorphic, in the obvious way: $(x \mapsto (1, x))$, to K, L, D respectively. Let $\hat{b}$ denote the image of $b \in Z(\mathcal{O}L)$ in $\mathcal{O}\hat{L}$ under the above isomorphism $L\tilde{=}\hat{L}$. Thus $\hat{b}$ is a $\hat{G}$-stable block idempotent of $\mathcal{O}\hat{L}$ and of $\mathcal{O}\hat{K}$ with defect group $Z(\hat{D})$ in $\mathcal{O}\hat{L}$ and defect group $\hat{D}$ in $\mathcal{O}\hat{K}$. Here $C_{\hat{G}}(\hat{D}) = i(\Omega) \times \hat{L}$ and $\hat{D}C_{\hat{G}}(\hat{D}) = i(\Omega) \times \hat{K}$, where i is described in Lemma 1.2(d).

Let $\hat{\gamma} = \gamma \circ i^{-1} : i(\Omega) \to \mathcal{O}^\times$ denote the linear character of $i(\Omega)$ corresponding to $\gamma : \Omega \to \mathcal{O}^\times$ and set $\hat{e} = \frac{1}{n}\sum_{\omega\in\Omega}\omega^{-1}(\omega, 1)$ so that $\hat{e}$ is the $\hat{G}$-stable block idempotent of $\mathcal{O}i(\Omega)$ corresponding to $\hat{\gamma}$. Thus $\hat{e}\hat{b}$ is a $\hat{G}$-stable block idempotent of $\mathcal{O}(\hat{D}C_{\hat{G}}(\hat{D}))$ with defect group $\hat{D}$ and $\hat{e}\hat{b}$ is also a block idempotent of $\mathcal{O}\hat{G}$ with defect group $\hat{D}$.

Let $\hat{\theta} \in \mathrm{Irr}_{\mathcal{K}}(\hat{b})$, where $\hat{b}$ is a block of $\hat{K}$ correspond to $\theta \in \mathrm{Irr}_{\mathcal{K}}(b)$ so that $\hat{D} \le Ker(\hat{\theta})$. Clearly $\hat{G} = \bigcup_{t\in\mathcal{T}}(i(\Omega) \times \hat{K})(1, t)$ and the union is disjoint.

Let $\hat{L}_1 = \{(1, \ell_1) \mid \ell_1 \in L_1\}$ and $\hat{K}_1 = \{(1, k_1) \mid k_1 \in K_1\}$, so that $L_1\tilde{=}\hat{L}_1$ and $K_1\tilde{=}\hat{K}_1$ canonically. Also $C_{\hat{K}}(A) = \hat{K}_1$, $C_{\hat{L}}(A) = \hat{L}_1$ and $W(\hat{b})$ is a block idempotent of $\mathcal{O}\hat{K}_1$ and of $\mathcal{O}\hat{L}_1$ with defect groups $\hat{D}$, $Z(\hat{D})$, respectively.

Since $\hat{c}(g_1^a, g_2^a) = \hat{c}(g_1, g_2)$ for all $g_1, g_1 \in G$ and all $a \in A$ by Lemma 2.11, A acts on the right on $\hat{G}$ according to: $(\omega, g)^a = (\omega, g^a)$ for all $\omega \in \Omega$, $g \in G$ and $a \in A$. Here $\hat{D} \le C_{\hat{G}}(A) = i(\Omega) \times C_G(A)$ and $C_{\hat{G}}(A) = \bigcup_{t\in\mathcal{T}}(i(\Omega) \times \hat{K}_1)(1, t)$, where the union is disjoint. Also $C_{\hat{G}}(A) \cap C_{\hat{G}}(\hat{D}) = i(\Omega) \times \hat{L}_1$ and $(\hat{D}C_{\hat{G}}(D)) \cap C_{\hat{G}}(A) = i(\Omega) \times \hat{K}_1$. Moreover $\widehat{W(b)}$ (the image of $W(b)$ under the isomorphism $L\tilde{=}\hat{L}$) is a $C_{\hat{G}}(A)$-stable block of $\mathcal{O}\hat{K}_1$ with defect group $\hat{D}$, $W(\hat{b}) = \widehat{W(b)}$ and $\hat{e}\widehat{W(b)}$ is a $C_{\hat{G}}(A)$-stable block idempotent of $i(\Omega) \times \hat{K}_1 = \hat{D}(C_{\hat{G}}(A) \cap C_{\hat{G}}(D))$ with defect group $\hat{D}$. Thus $\hat{e}\widehat{W(b)}$ is a block idempotent of $\mathcal{O}C_{\hat{G}}(A)$ with defect group $\hat{D}$ and $W(\hat{e}\hat{b})$ is also a block idempotent of $\mathcal{O}C_{\hat{G}}(A)$ with defect group $\hat{D}$ by [5, V, Lemma 3.10]. Set $\hat{\theta}_1 = \pi(\hat{K}, A)(\hat{\theta})$ so that $\hat{\theta}_1 \in \mathrm{Irr}_{\mathcal{K}}(W(\hat{b}))$.

Lemma 2.18. *Let $\hat{\pi}^*_{\mathcal{S}} : \hat{G} \to \mathcal{S}^\times$ be such that $(\omega, g) \mapsto \omega\pi^*_{\mathcal{S}}(g)$ for all $\omega \in \Omega$ and all $g \in G$ and set $\hat{\theta}^* = Tr_{\mathcal{S}} \circ \hat{\pi}^*_{\mathcal{S}} : \hat{G} \to \mathcal{O}$. Then*

(a) *$\hat{\pi}^*_{\mathcal{S}}$ is a group homomorphism with $\hat{D} \leq Ker(\hat{\pi}^*_{\mathcal{S}}) = Ker(\widehat{\theta^*})$;*

(b) *$\hat{\pi}_{\mathcal{S}} = \mathrm{Res}^{\hat{G}}_{\hat{K}}(\hat{\pi}^*_{\mathcal{S}}) : \hat{K} \to \mathcal{S}^\times$ is an irreducible representation of $\hat{K}$ over $\mathcal{K}$ and $\hat{\pi}^*_{\mathcal{S}}$ is an irreducible respresentation of $\hat{G}$ over $\mathcal{K}$ with character $\hat{\theta}^*$;*

(c) *$\hat{\theta}^*$ lies in the A-stable block $\hat{e}\hat{b}$ of $\mathcal{O}\hat{G}$ and $\hat{\theta}^*(\hat{x}) = 0$ for all $\hat{x} \in \hat{G}$ such that $\hat{x}\hat{D} \notin (\hat{G}/\hat{D})_{p'}$;*

(d) *$\mathrm{Res}^{\hat{G}}_{i(\Omega)\times\hat{K}}(\hat{\theta}^*) = \hat{\gamma} \times \hat{\theta}$ is an A-stable irreducible character that lies in the block $\hat{e}\hat{b}$ of $\mathcal{O}(i(\Omega) \times \hat{K})$ with defect group $\hat{D}$; and*

(e) *$\pi(\hat{G}, A)(\hat{\theta}^*) \in \mathrm{Irr}_{\mathcal{K}}(W(\hat{e}\hat{b}))$ and $\mathrm{Res}^{C_{\hat{G}}(A)}_{i(\Omega)\times\hat{K}}(\pi(\hat{G}, A)(\hat{\theta}^*)) = \hat{\gamma} \times \hat{\theta}_1$ is an irreducible character that lies in the block $\hat{e}W(\hat{b})$ of $\mathcal{O}(i(\Omega) \times \hat{K}_1)$ with defect group $\hat{D}$.*

Proof. Clearly Lemma 2.13 and [5, IV, Lemma 2.4] furnish a proof of (a)–(d). For (e), note that $\mathrm{Res}^{\hat{G}}_{i(\Omega)\times\hat{K}}(\hat{\theta}^*) = \hat{\gamma} \times \hat{\theta}$ and that $\pi(i(\Omega) \times \hat{K}, A)(\hat{\gamma} \times \hat{\theta}) = \hat{\gamma} \times \hat{\theta}_1$ is an irreducible character that lies in the block $\hat{e}W(\hat{b})$ of $\mathcal{O}(i(\Omega) \times \hat{K}_1)$ since $\pi(K, A)(G) = \theta_1$. Now [11, Theorem A(b)] implies that $\mathrm{Res}^{C_{\hat{G}}(A)}_{i(\Omega)\times\hat{K}_1}(\pi(\hat{G}, A)(\hat{\theta}^*)) = \hat{\gamma} \times \hat{\theta}_1$ and we are done.

Lemma 2.19. *Let $\widehat{\pi^*_{\mathcal{S}_1}} : C_{\hat{G}}(A) \to \mathcal{S}_1^\times$ be such that $(\omega, g) \mapsto \omega\pi^*_{\mathcal{S}_1}(g)$ for all $\omega \in \Omega$ and all $g \in C_G(A)$ and set $\widehat{\theta^*_1} = Tr_{\mathcal{S}_1} \circ \widehat{\pi^*_{\mathcal{S}_1}} : C_{\hat{G}}(A) \to \mathcal{O}$. Then*

(a) *$\widehat{\pi^*_{\mathcal{S}_1}}$ is a group homomorphism with $\hat{D} \leq Ker(\widehat{\pi^*_{\mathcal{S}_1}}) = Ker(\widehat{\theta^*_1})$;*

(b) *$\mathrm{Res}^{C_{\hat{G}}(A)}_{\hat{K}_1}(\widehat{\pi^*_{\mathcal{S}_1}}) = \hat{\pi}_{\mathcal{S}_1} : \hat{K}_1 \to \mathcal{S}_1^\times$ is an irreducible representation of $\hat{K}_1$ over $\mathcal{K}$ with character $\hat{\theta}_1$ and $\hat{\pi}^*_{\mathcal{S}_1}$ is an irreducible representation of $C_{\hat{G}}(A)$ over $\mathcal{K}$ with character $\hat{\theta}^*_1$;*

(c) *$\mathrm{Res}^{C_{\hat{G}}(A)}_{i(\Omega)\times\hat{K}_1}(\hat{\theta}^*_1) = \hat{\gamma} \times \hat{\theta}_1$;*

(d) *$\pi(\hat{G}, A)(\hat{\theta}^*) = \lambda\widehat{\theta^*_1}$ for a unique linear character λ of $C_{\hat{G}}(A)$ such that $i(\Omega) \times \hat{K}_1 \leq Ker(\lambda)$;*

(e) *$\hat{e}\widehat{W(b)} = W(\hat{e}\hat{b})$; and*

(f) *$\hat{\theta}^*_1 \in \mathrm{Irr}_{\mathcal{K}}(W(\hat{e}\hat{b}))$ and $\hat{\theta}^*_1(\hat{x}) = 0$ for all $\hat{x} \in C_{\hat{G}}(A)$ such that $\hat{x}\hat{D} \notin (C_{\hat{G}}(A)/\hat{D})_{p'}$.*

Proof. Clearly (a)–(c) hold and Lemma 2.18(e) yields (d). We have seen that $\hat{e}W(\hat{b})$ is a block idempotent with defect group $\hat{D}$ of both $\mathcal{O}(i(\Omega) \times \hat{K}_1)$ and $\mathcal{O}C_{\hat{G}}(A)$. Also $W(\hat{e}\hat{b})$ is a block idempotent of $\mathcal{O}C_{\hat{G}}(A)$ with defect group $\hat{D}$ that, by Lemma 2.18(e), covers the $C_{\hat{G}}(A)$-stable block $\hat{e}W(\hat{b})$ of $\mathcal{O}(i(\Omega) \times \hat{K}_1)$. Thus (e) holds by [5, V, Lemma 3.10]. By (c), $\hat{\theta}^*_1$ lies in a block of $\mathcal{O}C_{\hat{G}}(A)$ that covers the block $\hat{e}W(\hat{b})$, and again [5, V, Lemma 3.10] and the proof of Lemma 2.18(c) yield (f) .

Set $\bar{\hat{G}} = \hat{G}/\hat{D}$ and let $- : \hat{G} \to \bar{\hat{G}}$ denote the canonic group epimorphism. Clearly A induces an action on $\bar{\hat{G}}$ and $- : \bar{\hat{G}} \to \hat{G}/\hat{D}$ is an A-epimorphism. Here $|\bar{\hat{G}}| = n|G/D|$, $\bar{\hat{G}} = \bigcup_{t\in\mathcal{T}}(\overline{i(\Omega)} \times \bar{\hat{K}})\overline{(1,t)}$, where the union is disjoint and $C_{\bar{\hat{G}}}(A) = \overline{C_{\hat{G}}(A)} = C_{\hat{G}}(A)/\hat{D} = \bigcup_{t\in\mathcal{T}}\overline{(i(\Omega)} \times \bar{K}_1)\overline{(1,t)}$, where the union is also disjoint.

Since $\hat{D} \leq Ker(\hat{\pi}^*_{\mathcal{S}}) = Ker(\hat{\theta}^*)$, we can view $\hat{\pi}^*_s$ and $\hat{\theta}^*$ as lying in an A-stable block of defect 0 of $\hat{G}$ that is contained in the block $\hat{e}\hat{b}$ of $\hat{G}$ since $\hat{\theta}^*(1) = r$, where $r_p = |G/D|_p = |\hat{G}/\hat{D}|_p$. Similar statements hold for $\pi^*_{\mathcal{S}_1}, \hat{\theta}^*_1,\ G_{\bar{\hat{G}}}(A)$ and $C_{\hat{G}}(A)$.

Remark 2.20. Let $(\omega, kt) \in \hat{G} = \Omega \times G$, where $\omega \in \Omega,\ k \in K,\ t \in \mathcal{T}$ and let $kt = (kt)_p(kt)_{p'}$. Then $(kt)_p \in K$ since $|G/K|_p = 1$ and $(\omega, (kt)_{p'}) \in \hat{G}_{p'}$. Thus $(\omega, kt)_p = (1, (kt)_p)$, $(\omega, kt)_{p'} = (\omega, (kt)_{p'})$. It follows that $(\omega, kt)\hat{D} \in (\bar{\hat{G}})_{p'}$ if and only if $(kt)D \in (G/D)_{p'}$. A similar result holds for $(\omega, k_1 t) \in C_{\hat{G}}(A) = \Omega \times C$, where $\omega \in \Omega,\ k_1 \in K_1,\ t \in \mathcal{T}$.

As in Lemma 2.19(d), let λ be the unique linear character of $C_{\hat{G}}(A)$ with $i(\Omega) \times \hat{K}_1 \leq Ker(\lambda)$ such that $\pi(\hat{G}, A)(\widehat{\theta^*}) = \lambda\hat{\theta}^*_1$. Since $\hat{G} = \hat{K}C_{\hat{G}}(A)$, there is a unique linear character $\hat{\lambda}$ of $\hat{G}$ such that $i(\Omega) \times \hat{K} \leq Ker(\hat{\lambda})$ and $\mathrm{Res}^{\hat{G}}_{C_{\hat{G}}(A)}(\hat{\lambda}) = \lambda$. It follows that $\hat{\lambda}^{-1}\hat{\theta}^* \in \mathrm{Irr}_{\mathcal{K}}(\hat{e}\hat{b})^A$ and $\pi(\hat{G}, A)(\hat{\lambda}^{-1}\hat{\theta}^*) = \hat{\theta}^*_1$. At this point, for each $t \in \mathcal{T}$, we replace w_t by $\hat{\lambda}(t)w_t$ and v_t by $\hat{\lambda}(t)v_t$ in Proposition 2.1. Then, since $w_t = s_t\alpha_t$ where $s_t \in \mathcal{S}^\times$ and $\alpha_t \in b + \mathcal{I}(\mathcal{O}L)b)$, s_t is replaced by $\lambda(t)s_t$ for each $t \in \mathcal{T}$. With this replacement, we obtain the new $\mathcal{O}$-linear map $\tilde{\pi}^*_{\mathcal{S}} : \mathcal{O}G \to \mathcal{S}$ sending kt to $\pi_{\mathcal{S}}(kb)\hat{\lambda}(t)^{-1}s_t^{-1}$ so that $Tr_{\mathcal{S}}(\pi^*_{\mathcal{S}}(kt)) = (\hat{\lambda}^{-1}\hat{\theta}^*)(kt)$ for all $k \in K$ and $t \in \mathcal{T}$. Consequently after replacement, we may assume that

$$\pi(\hat{G}, A)(\hat{\theta}^*) = \hat{\theta}^*_1. \tag{2.8}$$

We have set

$$\mathcal{B} = C_{(\mathcal{O}G)b}(\mathcal{S}) = \oplus_{t\in\mathcal{T}}(((\mathcal{O}D)b)v_t) = \oplus_{\substack{t\in\mathcal{T}\\ d\in D}}\mathcal{O}(db)v_t,$$

so that $\mathcal{B}$ can be viewed as an $\mathcal{N} = D \rtimes (H_1/L_1)$-twisted group $\mathcal{O}$-algebra with $\mathcal{O}$-bases $\{(db)v_t | d \in D,\ tL_1 \in H_1/L_1\}$, where

$$((d_1 b)v_{t_1})((d_2 b)v_{t_2}) = c(t_1L_1, t_2L_1)(d_1({}^{t_1}d_2))v_{t_3}$$

if $t_1t_2 \in t_3L_1$ for a unique $t_3 \in \mathcal{T}$ and $\mathcal{B}_{(d,tL_1)} = \mathcal{O}(db)v_t$ for all $d \in D$ and $t \in \mathcal{T}$. Also we have set $\mathcal{B}' = C_{(\mathcal{O}C)W(b)}(\mathcal{S}_1) = \oplus_{t\in\mathcal{T}}(((\mathcal{O}D)W(b))v'_t)$ and we view $\mathcal{B}'$ as an $\mathcal{N} = D \rtimes (H_1/L_1)$-twisted group $\mathcal{O}$-algebra, where $\mathcal{B}'_{(d,tL_1)} = \mathcal{O}(dW(b))v'_t$ for all $d \in D$ and $t \in \mathcal{T}$. Thus $\Phi : \mathcal{B}' \to \mathcal{B}$ is an $\mathcal{N}$-graded $\mathcal{O}$-algebra isomorphism sending $(dW(b))v'_t$ to $(db)v_t$ for all $d \in D$ and $t \in \mathcal{T}$.

We inflate $c \in Z^2(H_1/L_1, \mathcal{O}^\times)$ to an element $\tilde{c} \in Z^2(\mathcal{N}, \mathcal{O}^\times)$, where

$$\tilde{c}((d_1, t_1L_1), (d_2, t_2L_1)) = c(t_1L_1, t_2L_2)$$

for all $d_1, d_2 \in D$ and all $t_1, t_2 \in \mathcal{T}$. Then, using $\tilde{c}$ and Lemma 1.2, we obtain a finite group $\tilde{\mathcal{N}} = \Omega\tilde{\times}\mathcal{N}$, where $|\tilde{\mathcal{N}}| = n|D|m$ and we let A act trivially on $\tilde{\mathcal{N}}$. Here $\tilde{D} = \{(1, (d, L_1)) | d \in D\}$ is a normal Sylow p-subgroup of $\tilde{\mathcal{N}}$, $D \tilde{=} \tilde{D}$ via the map $d \mapsto (1, (d, L_1))$ for all $d \in D$, $\{(\omega, (1, tL_1)) | \omega \in \Omega,\ t \in \mathcal{T}\}$ is a complement to D and $\tilde{\mathcal{N}} = \bigcup_{t\in\mathcal{T}}(i(\Omega) \times \tilde{D})(1, (1, tL_1))$, where the union is disjoint.

Set $\tilde{e} = \frac{1}{n}\sum \omega^{-1}(\omega, (1, L_1)) \in \mathcal{O}i(\Omega)$, so that $(\omega, (d, tL_1))\tilde{e} = \omega(1, (d, tL_1))\tilde{e}$ for all $\omega \in \Omega,\ d \in D$ and $t \in \mathcal{T}$. Then $\tilde{e}$ is an $\tilde{\mathcal{N}}$-stable block idempotent of $\mathcal{O}i(\Omega)$ (corresponding to $\tilde{\gamma} = \gamma \circ i^{-1}$), $\tilde{e} \in Z(\mathcal{O}\tilde{\mathcal{N}})$, $(\mathcal{O}\tilde{\mathcal{N}})\tilde{e} = \oplus_{\substack{d\in D\\ t\in\mathcal{T}}}\mathcal{O}(1, (d, tL_1))\tilde{e}$ in $\mathcal{O}$-mod and the $\mathcal{O}$-linear map

$$\Psi : (\mathcal{O}\tilde{\mathcal{N}})\tilde{e} \to \mathcal{B}' = \oplus_{\substack{d\in D\\ t\in\mathcal{T}}}\mathcal{O}((dW(b))v'_t)$$

such that $(1,(d,tL_1))\tilde{e} \mapsto (dW(b))v'_t$ for all $d \in D$ and $\in \mathcal{T}$ is an $\mathcal{O}$-algebra isomorphism.

Let A act diagonally on the right on $\bar{\hat{G}} \times \tilde{\mathcal{N}}$ so that $C_{\bar{\hat{G}}\times\mathcal{N}}(A) = C_{\bar{\hat{G}}}(A) \times \tilde{\mathcal{N}}$. Set $\Delta = \{(\overline{(\omega,kt)},(\omega^{-1},(d,tL_1)))|\omega \in \Omega,\ d \in D, k \in K \text{ and } t \in \mathcal{T}\}$.

The following result is easily verified:

Lemma 2.21. (a) *Δ is an A-invariant subgroup of $\bar{\hat{G}} \times \tilde{\mathcal{N}}$ with $|\Delta| = n|G|$ and $C_\Delta(A) = \{(\overline{(\omega,k_1t)},(\omega^{-1},(d,tL_1)))|\omega \in \Omega,\ d \in D,\ t \in \mathcal{T}$ and $k_1 \in K_1\}$;*

(b) *$(i(\Omega)\times i(\Omega))\cap\Delta = \{(\overline{(\omega,1)},(\omega^{-1},(1,L_1)))|\omega \in \Omega\} \leq Z(\Delta)\cap C_\Delta(A)$ and the map of $\Omega \to (i(\Omega)\times i(\Omega))\cap\Delta$ such that $\omega \mapsto (\overline{(\omega,1)},(\omega^{-1},(1,L_1)))$ for all $\omega \in \Omega$ is an isomorphism; and*

(c) *$\mathcal{D} = \{(\overline{(1,1)},(1,(d,L_1)))|d \in D\} \leq \Delta$ and the map of $D \to \mathcal{D}$ such that $d \mapsto (\overline{(1,1)},(1,(d,L_1)))$ for all $d \in D$ is an isomorphism.*

Recall the $\mathcal{O}$-algebra isomorphism $\Phi : \mathcal{B}' \to \mathcal{B}$ and $\Psi : (\mathcal{O}\tilde{\mathcal{N}})\tilde{e} \to \mathcal{B}'$. Thus $\mathrm{Irr}_\mathcal{K}((\mathcal{K}\tilde{\mathcal{N}})\tilde{e}) = \{\psi\Phi\Psi|\psi \in \mathrm{Irr}_\mathcal{K}(\mathcal{K}\mathcal{B})\}$. Also $\hat{\theta}^* = Tr_\mathcal{S} \circ \hat{\pi}^*_\mathcal{S} \in \mathrm{Irr}_\mathcal{K}(\hat{G})$ and $\hat{\theta}^*_1 = Tr_{\mathcal{S}_1} \circ \hat{\pi}^*_{\mathcal{S}_1} \in \mathrm{Irr}_\mathcal{K}(C_{\hat{G}}(A))$. Let $\bar{\hat{\theta}}^*$ and $\bar{\hat{\theta}}^*_1$ denote the irreducible characters of $\bar{\hat{G}} = \hat{G}/\hat{D}$ and of $C_{\bar{\hat{G}}}(A) = C_{\hat{G}}(A)/\hat{D}$ from which $\hat{\theta}^*$ and $\hat{\theta}^*_1$ are inflated, respectively. Thus $\bar{\hat{\theta}}^* \times (\psi\Phi\Psi) \in \mathrm{Irr}_\mathcal{K}(\bar{\hat{G}}\times\tilde{\mathcal{N}})$ and $\bar{\hat{\theta}}^*_1 \times (\psi\Phi\Psi) \in \mathrm{Irr}_\mathcal{K}(C_{\bar{\hat{G}}\times\tilde{\mathcal{N}}}(A))$ and $\Delta\cap i(\Omega) \leq Ker(\bar{\hat{\theta}}^* \times (\psi\Phi\Psi)) \cap Ker(\bar{\hat{\theta}}^*_1 \times (\psi\Phi\Psi))$ for all $\psi \in \mathrm{Irr}_\mathcal{K}(\mathcal{K}\mathcal{B})$.

Let $\omega \in \Omega,\ t \in \mathcal{T},\ d \in D,\ k \in K$ and $k_1 \in K_1$ and $\psi \in \mathrm{Irr}_\mathcal{K}(\mathcal{B})$. Note that $\overline{(\omega,kt)} \in \bar{\hat{G}}_{p'}$ if and only if $k_1tD \in (G/D)_{p'}$ and $\overline{(\omega,k_1t)} \in C_{\bar{\hat{G}}}(A)_{p'}$ if and only if $k_1tD \in (C_G(A)/D)_{p'}$ by Remark 2.20. Also $(\overline{(\omega,kt)},(\omega^{-1},(d,tL_1))) \in \Delta$ and $(\overline{(\omega,k_1t)},(\omega^{-1},(d,tL_1))) \in C_\Delta(A)$.

Thus we have

$$(\bar{\hat{\theta}}^* \times (\psi\Phi\Psi))(\overline{(\omega,kt)},\ (\omega^{-1},(d,tL_1))) = \begin{cases} 0 & \text{if } ktD \notin (G/D)_{p'}, \\ (Tr_{\mathcal{S}}\circ\pi^*_\mathcal{S})((kt)b)\psi((db)v_t) & \text{if } ktD \in (G/D)_{p'}; \end{cases} \tag{2.9}$$

and

$$(\bar{\hat{\theta}}^*_1 \times (\psi\Phi\Psi))(\overline{(\omega,k_1t)},(\omega^{-1},(d,tL_1))) = \begin{cases} 0 & \text{if } k_1tD \notin (C/D)_{p'}, \\ (Tr_{\mathcal{S}_1^\circ}\pi^*_{\mathcal{S}_1})((k_1t)W(b))\psi((db)v_t) & \text{if } k_1tD \in (C/D)p'. \end{cases} \tag{2.10}$$

Consequently

(2.11) if $(kt)D \in (G/D)_{p'}$, then

$$\begin{aligned} &\sum_{d\in D}((\bar{\hat{\theta}}^* \times (\psi\Phi\Psi))(\overline{(\omega,kt)},(\omega^{-1},(d,tL_1))) \\ &\cdot((\bar{\hat{\theta}}^* \times \psi\Phi\Psi)((\overline{(\omega,kt)},(\omega^{-1},dtL_1))^{-1})) \\ &= ((Tr_{\mathcal{S}}\circ\pi^*_\mathcal{S})((kt)b))((Tr_{\mathcal{S}}\circ\pi^*_\mathcal{S})(((kt)b)^{-1})) \\ &\cdot|D : C_D(t)| \sum_{d\in C_D(t)}(\psi((db)v_t))(\psi(((db)v_t)^{-1})); \end{aligned}$$

(2.12) if $k_1tD \in (C/D)_{p'}$, then

$$\begin{aligned}&\sum_{d\in D}((\bar{\hat{\theta}}_1^* \times \psi\Phi\Psi)(\overline{(\omega, k_1t)}, (\omega^{-1}, (d, tL_1)))\\ &\cdot((\bar{\hat{\theta}}_1^* \times \psi\Phi\Psi)(\overline{(\omega, k_1t)}, (\omega^{-1}, (d, tL_1))^{-1}))\\ &= ((Tr_{\mathcal{S}_1^\circ}\pi^*_{\mathcal{S}_1})((k_1t)w(b)))((Tr_{\mathcal{S}_1^0}\pi^*_{\mathcal{S}_1})(((k_1t)W(b))^{-1}))\\ &\cdot|D : C_D(t)| \sum_{d\in C_D(t)}(\psi(db)v_t)(\psi(((db)v_t)^{-1}));\end{aligned}$$

and

$$\begin{aligned}&\sum_{d\in D}((\bar{\hat{\theta}}^* \times (\psi\Phi\Psi))(\overline{(\omega, k_1t)}, (\omega^{-1}, (d, tL_1)))\\ &\cdot((\bar{\hat{\theta}}_1^* \times \psi\Phi\Psi))((\overline{(\omega, k_1t)}, (\omega^{-1}, (d, tL_1))^{-1}))\\ &= ((Tr_{\mathcal{S}^0}\pi^*_{\mathcal{S}})((k_1t)b))((Tr_{\mathcal{S}_1^0}\pi_{\mathcal{S}_1})(((k_1t)W(b))^{-1}))\\ &\cdot|D : C_D(t)| \sum_{d\in C_D(t)}(\psi((db)v_t))(\psi(((db)v_t)^{-1})).\end{aligned} \tag{2.13}$$

Now compare (2.3), (2.4), (2.5), (2.6), (2.9), (2.10), (2.11) and (2.12).

Thus we conclude that $(\mathrm{Res}_{\Delta}^{\bar{\hat{G}}\times\tilde{\mathcal{N}}}(\bar{\hat{\theta}}^* \times (\psi\Phi\Psi)),\ \mathrm{Res}_{\Delta}^{\hat{G}\times\mathcal{N}}(\bar{\hat{\theta}}^* \times (\psi\Phi\Psi)))_\Delta = 1$ so that $\mathrm{Res}_{\Delta}^{\bar{\hat{G}}\times\tilde{\mathcal{N}}}(\bar{\hat{\theta}}^* \times (\psi\Phi\Psi)) \in \mathrm{Irr}_{\mathcal{K}}(\Delta)$. Similarly $\mathrm{Res}_{C_\Delta(A)}^{C_{\bar{\hat{G}}}(A)\times\tilde{\mathcal{N}}}(\bar{\hat{\theta}}_1^* \times (\psi\Phi\Psi)) \in \mathrm{Irr}_{\mathcal{K}}(C_\Delta(A))$.

Since we have assured that $\pi(\hat{G}, A)(\hat{\theta}^*) = \hat{\theta}_1^*$, we conclude that $\pi(\bar{\hat{G}}, A)(\bar{\hat{\theta}}^*) = \bar{\hat{\theta}}_1^*$. Then [11, Theorem A(b)] (with $H = \Delta$) implies that

$$\begin{aligned}\pi(\Delta, A)(\mathrm{Res}_{\Delta}^{\bar{\hat{G}}\times\tilde{\mathcal{N}}}(\bar{\hat{\theta}}^* \times (\psi\Phi\Psi))) &= \mathrm{Res}_{C_\Delta(A)}^{C_{\bar{\hat{G}}}(A)\times\tilde{\mathcal{N}}}(\pi(\bar{\hat{G}} \times \tilde{\mathcal{N}}, A)(\bar{\hat{\theta}}^* \times (\psi\Phi\Psi)))\\ &= \mathrm{Res}_{C_\Delta(A)}^{C_{\bar{\hat{G}}}(A)\times\tilde{\mathcal{N}}}(\bar{\hat{\theta}}_1^* \times (\psi\Phi\Psi)).\end{aligned}$$

Hence $\rho = (\mathrm{Res}_{C_\Delta(A)}^{\bar{\hat{G}}\times\tilde{\mathcal{N}}}(\bar{\hat{\theta}}^* \times (\psi\Phi\Psi)),\ \mathrm{Res}_{C_\Delta(A)}^{C_{\bar{\hat{G}}}(A)\times\tilde{\mathcal{N}}}(\bar{\hat{\theta}}_1^* \times (\psi\Phi\Psi)))_{C_\Delta(A)}$ is relatively prime to q. However (2.7) and (2.13) imply that $\rho = (\mathrm{Res}_C^G(\theta_\psi),\ \theta_{1\psi})_C$. Consequently $\pi(G, A)(\theta_\psi) = \theta_{1\psi}$ by [10, Theorem 13.1(c)] which concludes our proof of Theorem 2.

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